16.4: Green's Theorem

Julia Jackson

Department of Mathematics
The University of Oklahoma

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Overview

Over the last two sections, we have been learning to evaluate line integrals of vector fields:

$$\int_C \vec{F} \cdot d\vec{r}$$

In 16.2, we learned to evaluate such integrals by utilizing a parametrization of the path of integration. Then in 16.3 we learned that if $\vec{F}$ is conservative, the Fundamental Theorem of Line Integrals provides a useful alternative method for evaluating such integrals.

This raises a question: is there an alternative way to evaluate line integrals of *nonconservative* vector fields? It would be nice to such a method, to grant us flexibility similar to what we had for line integrals of conservative vector fields.

In this section we introduce Green's Theorem, a powerful method of integration that converts line integrals of vector fields on $\mathbb{R}^2$ along piecewise smooth, simple, closed curves into double integrals.

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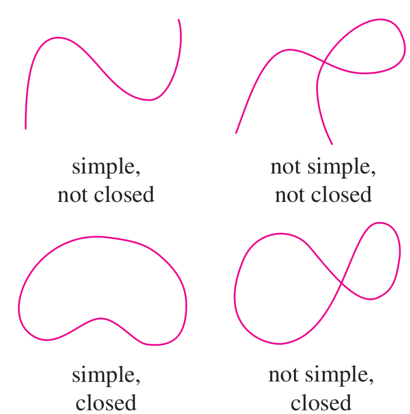
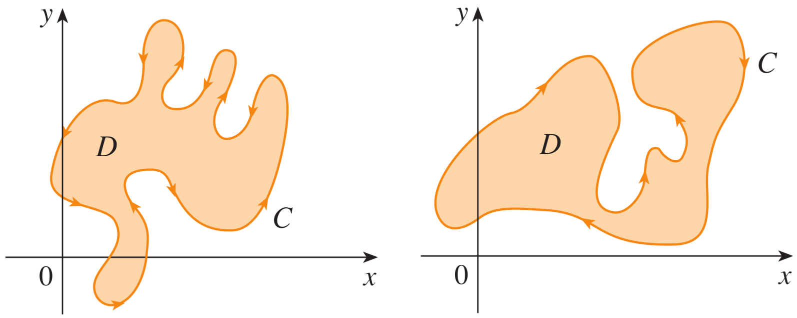
Simple and Closed Curves

Before we get to the theorem, we need some definitions. A curve C in $\mathbb{R}^2$ is said to be **closed** if its initial and terminal points coincide. The canonical examples are loops or circles. C is called **simple** if it does not intersect itself, except perhaps at its endpoints. Here are some examples of each:



Orientation

Another definition: we say a simple, closed curve C has **positive orientation** if it is drawn counterclockwise, i.e. if its interior D is always on its left as C is drawn. Otherwise, we say that C has **negative orientation**:



(a) Positive orientation (b) Negative orientation

One Last Remark

One last thing: Recall that in section 16.2 we saw that if we have a vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$, then:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$$

Many of the problems in this section of the text will be stated in the form on the right, so remember this connection. With this in mind, we state Green's Theorem.

Green's Theorem

Theorem (Green): Let C be a positively-oriented, piecewise-smooth, simple, closed curve in $\mathbb{R}^2$, and let D be its interior. If $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ and $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on D, then

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A$$

Nonconservative Vector Fields

Notice that the statement of Green's theorem didn't specify that $F(x, y)$ must be non-conservative, and yet I stated in the introduction to this section that, in practice, we typically only use Green's Theorem when $\vec{F}(x, y)$ is a non-conservative vector field. Why is this so?

Well, suppose that C is *any* closed curve in $\mathbb{R}^2$ starting and ending at (a, b), and that $\vec{F}(x, y) = \nabla f(x, y)$ is a conservative vector field. Then by the Fundamental Theorem for Line Integrals:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = f(a, b) - f(a, b) = 0$$

There's no need to call up Green's Theorem at all, and in fact, Green's theorem only makes this problem look far more complicated than it needs to be.

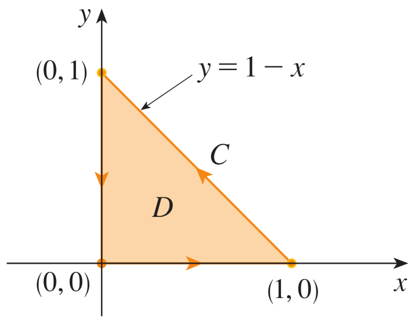
Example

Evaluate

$$I_1 = \int_C x^4 \, \mathrm{d}x + xy \, \mathrm{d}y$$

where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$; from $(1,0)$ to $(0,1)$; and from $(0,1)$ to $(0,0)$.

Let's begin by drawing C:



Example, cont.

Now, if we wanted to evaluate I_1 using the methods of §16.2, we would have to parametrize three separate paths and compute three separate line integrals! But note: C is a positively-oriented, simple, closed curve, consisting of three smooth pieces. So, Green's Theorem applies! Letting $P(x,y) = x^4$ and $Q(x,y) = xy$, we have:

$$\begin{aligned}\int_C x^4 \, \mathrm{d}x + xy \, \mathrm{d}y &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A \\ &= \iint_D (y - 0) \, \mathrm{d}A \\ &= \int_0^1 \int_0^{1-x} y \, \mathrm{d}y \, \mathrm{d}x = \boxed{\frac{1}{6}}\end{aligned}$$

That was much easier! (By the way, this is also the line integral along C of $\vec{F}(x,y) = \langle x^4, xy \rangle$.)

Notation

A quick note: if the piecewise-smooth, simple, closed curve C is described without giving its orientation, we often write

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \oint_C P(x,y) \, \mathrm{d}x + Q(x,y) \, \mathrm{d}y$$

or

$$\ointctrclockwise_C \vec{F} \cdot \mathrm{d}\vec{r} = \ointctrclockwise_C P(x,y) \, \mathrm{d}x + Q(x,y) \, \mathrm{d}y$$

to indicate that the positive orientation of C should be used to calculate the integral.

Flowchart

To summarize the last three sections: to evaluate $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$:

1. Is $\vec{F}(x,y)$ conservative?
   - 1.1 Yes: Try using the Fundamental Theorem
   - 1.2 No: Continue
2. Do the conditions for Green's Theorem hold?
   - 2.1 Yes: Try using Green's theorem
   - 2.2 No: Continue
3. Use the direct methods of §16.2 (parametrize C, etc.).

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Exercises

1. Evaluate
$$J_1 = \oint_{C_1} (3y - e^{\sin(x)})\,\mathrm{d}x + \left(7x + \sqrt{y^4+1}\right)\mathrm{d}y$$
where C_1 is the circle $x^2 + y^2 = 9$.
2. Evaluate
$$J_2 = \oint_{C_2} y^2\,\mathrm{d}x + 3xy\,\mathrm{d}y$$
where C_2 is the boundary of the semiannular region D_2 in the upper half-plane between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
3. Evaluate
$$J_3 = \int_{C_3} \vec{F}(x,y) \cdot \mathrm{d}\vec{r}$$
where $\vec{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and C_3 is the square with vertices $(-1,-1)$, $(1,-1)$, $(1,1)$, and $(-1,1)$, traced counterclockwise.
4. How can you apply Green's Theorem to *negatively-oriented*, piecewise-smooth, simple, closed curves?

Solutions

1. $J_1 = \int_0^{2\pi} \int_0^3 4r\,\mathrm{d}r\,\mathrm{d}\theta = \boxed{36\pi}$
2. $J_2 = \int_0^{\pi} \int_1^2 r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta = \boxed{\frac{14}{3}}$
3. $J_3 = \int_{C_3} \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y = \iint_{D_3} 0\,\mathrm{d}A = \boxed{0}$, where D_3 is the region enclosed by the square C_3.
4. Recall that for line integrals, we have:
$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = -\int_{-C} \vec{F} \cdot \mathrm{d}\vec{r}$$
where $-C$ is just C with the opposite orientation. If C is negatively-oriented, simply apply Green's Theorem to the integral on the right.