16.3: The Fundamental Theorem for Line Integrals

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Overview

In the previous section, we learned how to evaluate line integrals of vector fields over curves C:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r}$$

In this section, we will show that if $\vec{F}(\vec{x})$ is a gradient field ∇f (i.e. if $\vec{F}(\vec{x})$ is conservative), then there is a very efficient way to evaluate the integral above. We will then turn our attention to figuring out how to determine if $\vec{F}(\vec{x})$ is conservative.

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The Fundamental Theorem

Recall that a vector field $\vec{F}(\vec{x})$ is called *conservative* if $\vec{F}(\vec{x}) = \nabla f$ for some real-valued function f, i.e., if $\vec{F}(\vec{x})$ is a gradient field. We call f a *potential function* of $\vec{F}(\vec{x})$.

Theorem (The Fundamental Theorem for Line Integrals): Let C be a smooth curve (or a concatenation of a finite number of smooth curves) in $\mathbb{R}^2$ with initial point (a, b) and terminal point (s, t). Let $\vec{F}(x, y)$ be a conservative vector field with potential function $f(x, y)$ which is continuous on C. Then:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \nabla f \cdot \mathrm{d}\vec{r} = f(s, t) - f(a, b)$$

An analogous result holds for vector fields on $\mathbb{R}^3$.

Why "Fundamental Theorem"?

This is called the fundamental theorem because it looks very much like the Fundamental Theorem of Calculus: we evaluate $\int_C \nabla f \cdot \mathrm{d}\vec{r}$ by evaluating something like an antiderivative of ∇f at the endpoints of the curve C.

Example

Let $\vec{F}(x,y) = \langle 2x, 1\rangle$. Note that $f(x,y) = x^2 + y - 2$ is a potential function for $\vec{F}(x,y)$, as $\nabla f(x,y) = \langle 2x, 1\rangle$. Let C_1 be the top half of the circle $x^2+y^2=1$, traced counterclockwise; let C_2 be the bottom half of the same circle, traced clockwise; and let C_3 be the straight-line path between the initial and terminal points of these semicircles. Evaluate $\int_C \vec{F}\cdot \mathrm{d}\vec{r}$, for $C = C_1$, C_2, and C_3, respectively.

Note that each of the paths above begins at $(1,0)$ and ends at $(-1,0)$. Therefore, by the theorem on the previous slide, we have:

$$\begin{aligned}\int_C \vec{F}\cdot \mathrm{d}\vec{r} &= \int_C \nabla f \cdot \mathrm{d}\vec{r} \\ &= f(-1,0) - f(1,0) = (1+0-2) - (1+0-2) = \boxed{0}\end{aligned}$$

regardless of which path we choose for C.

Key Observation

In the previous example, we saw an instance of the following result:

Theorem: Let (a,b) and (r,s) be two points in $\mathbb{R}^2$. If $\vec{F}(x,y)$ is a continuous, conservative vector field on $\mathbb{R}^2$, then $\int_C \vec{F}\cdot \mathrm{d}\vec{r}$ is independent of the path C chosen between (a,b) and (r,s). An analogous result holds for vector fields on $\mathbb{R}^3$.

We summarize this by saying that line integrals of conservative vector fields are independent of path.

The Converse

In fact, it turns out that line integrals of a vector field $\vec{F}(\vec{x})$ are independent of path *only if* $\vec{F}(\vec{x})$ is conservative.

Thus, conservative vector fields are quite special! So... how can we tell if a vector field is conservative? That is the question we set about answering next.

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Is $\vec{F}(\vec{x})$ Conservative?

How can we decide if a vector field is conservative?

Well, the most direct way is to see if we can find a real-valued function f such that $\vec{F}(\vec{x}) = \nabla f$. But this is not very efficient. After all, if, for example, $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ *is* conservative, then takes time to find a function $f(x, y)$ with $f_x(x, y) = P(x, y)$ and $f_y(x, y) = Q(x, y)$. But if $\vec{F}(\vec{x})$ *isn't* conservative, then we need to show that such an $f(x, y)$ cannot exist, and it's not immediately obvious how to go about doing so. After all, it could certainly be that such an $f(x, y)$ does exist, but we're just not very good at finding it.

It would be nice if there were a quicker, more sensitive test that could tell us immediately if a vector field is conservative. And there is! For now, we will develop a test for vector fields on $\mathbb{R}^2$.

Toward a Test

Suppose that $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is conservative, i.e. we actually do have a real-valued function $f(x, y)$ such that:

$$f_x(x, y) = P(x, y) \quad f_y(x, y) = Q(x, y)$$

Recall that Clairaut's theorem tells us that:

$$P_y(x, y) = f_{xy}(x, y) = f_{yx}(x, y) = Q_x(x, y)$$

A Result

Therefore, we have the following theorem:

Theorem: If $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is conservative, then

$$P_y(x, y) = Q_x(x, y)$$

A handy corollary to this result is that if $P_y(x, y) \neq Q_x(x, y)$ then $\vec{F}(x, y)$ *cannot* be conservative! This gives us a nice test that tells us when a vector field is *not* conservative.

Example

Show that the vector field $\vec{F}(x, y) = \langle x - y, x - 2 \rangle$ is not conservative.

Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$. Then:

$$P_y(x, y) = -1 \quad \text{and} \quad Q_x(x, y) = 1$$

Since these are not equal, $\vec{F}(x, y)$ is not conservative.

The Converse

We now have a way to show that a vector field is not conservative. That's nice, but how can we show that a vector field *is* conservative?

Well, it turns out that the converse of the theorem above is true (with some technical conditions we might discuss later). In other words, if $\vec{F}(x, y) = \big\langle P(x, y), Q(x, y) \big\rangle$ has

$$P_y(x, y) = Q_x(x, y)$$

then $\vec{F}(x, y)$ is conservative.

Therefore, we in fact have a single test that tells us if a vector field is or is not conservative.

Example

Is the vector field $\vec{G}(x, y) = \big\langle 3 + 2xy, x^2 - 3y^2 \big\rangle$ conservative?

Let $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$. We have:

$$P_y(x, y) = 2x \quad \text{and} \quad Q_x(x, y) = 2x$$

Since these are the same, $\vec{G}(x, y)$ $\boxed{\text{is conservative!}}$

Finding a Potential Function

We now have a quick way of determining whether a vector field $\vec{F}(x, y)$ is conservative. Recall that we wanted such a test to help us determine whether the fundamental theorem of line integrals

$$\int_C \vec{F} \cdot d\vec{r} = f(s, t) - f(a, b)$$

applies to $\vec{F}(\vec{x})$.

If we wish to evaluate $\int_C \vec{F} \cdot d\vec{r}$ and we have established that $\vec{F}(x, y)$ is conservative, the next thing we need is a potential function $f(x, y)$ for $\vec{F}(x, y)$. How can we find one?

Example

Find a potential function $g(x, y)$ for the conservative vector field $\vec{G}(x, y) = \langle 3 + 2xy, x^2 - 3y^2 \rangle$ from the previous example.

We want a function $g(x, y)$ such that $\nabla g(x, y) = \vec{G}(x, y)$, i.e. $g_x(x, y) = 3 + 2xy$ and $g_y(x, y) = x^2 - 3y^2$.

First, since $g_x(x, y) = 3 + 2xy$, we have:

$$g(x, y) = 3x + x^2 y + f(y)$$

where $f(y)$ is some function of y (as this is just 0 when we calculate $g_x(x, y)$).

Example, cont.

Differentiating the $g(x, y)$ we just found with respect to y and comparing to $\vec{G}(x, y)$, we have:

$$g_y(x, y) = x^2 + f'(y) = x^2 - 3y^2$$

so that $f'(y) = -3y^2$, i.e. $f(y) = -y^3 + K$, for any constant K.

Therefore, a potential function for $\vec{G}(x, y)$ is:

$$g(x, y) = 3x + x^2 y + f(y) = \boxed{3x + x^2 y - y^3}$$

You may verify this by computing the gradient of $g(x, y)$!

What About Vector Fields on $\mathbb{R}^3$?

We will develop a convenient test for detecting vector fields on $\mathbb{R}^3$ later. For now, unfortunately, we don't have one.

Of course, we always have the most basic test: $\vec{F}(x, y, z)$ is conservative if we can find a potential function $f(x, y, z)$ for it. Let's see how to find a potential function of a vector field on $\mathbb{R}^3$.

Example

Assume that $\vec{F}(x, y, z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$ is conservative. Find a potential function $f(x, y, z)$ for $\vec{F}(x, y, z)$.

First, we must have:

$$f_x(x, y, z) = y^2$$

Which means that

$$f(x, y, z) = xy^2 + g(y, z)$$

for some function $g(y, z)$ (as this becomes 0 when we compute $f_x(x, y, z)$).

Example, cont.

From this and $\vec{F}(x,y,z)$, we have:

$$f_y(x,y,z) = 2xy + e^{3z} = 2xy + g_y(y,z)$$

so that

$$g_y(y,z) = e^{3z}$$

which means that

$$g(y,z) = ye^{3z} + h(z)$$

where $h(z)$ is some function of z (as this becomes 0 when we compute $g_y(y,z)$). Therefore, combining with our work above we have:

$$f(x,y,z) = xy^2 + ye^{3z} + h(z)$$

Example, cont.

Finally, from this and $\vec{F}(x,y,z)$ we have:

$$f_z(x,y,z) = 3ye^{3z} = 3ye^{3z} + h'(z)$$

so that

$$h'(z) = 0$$

which means that

$$h(z) = K$$

for any constant K. Thus, a potential function $f(x,y,z)$ for $\vec{F}(x,y,z)$ is:

$$\begin{aligned} f(x,y,z) &= xy^2 + g(y,z) \\ &= xy^2 + ye^{3z} + h(z) = \boxed{xy^2 + ye^{3z}} \end{aligned}$$

(you may verify this by checking that $\vec{F}(x,y,z) = \nabla f(x,y,z)$).

Summary

Here are the key results we learned:

1. If $\vec{F}(x,y)$ is a conservative vector field on $\mathbb{R}^2$, i.e. if $\vec{F}(x,y) = \nabla f(x,y)$ for some real-valued function $f(x,y)$, then $\int_C \vec{F} \cdot d\vec{r} = f(s,t) - f(a,b)$ for any path C starting at (a,b) and ending at (s,t). An analogous result holds on $\mathbb{R}^3$.
2. This result *only* holds for conservative vector fields, no others.
3. A vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ on $\mathbb{R}^2$ is conservative if and only if $P_y(x,y) = Q_x(x,y)$.

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1. Is the vector field $\vec{F}(x,y) = \langle y^2 - 2x, 2xy \rangle$ conservative? If so, find a potential function $f(x,y)$ for $\vec{F}(x,y)$.
2. Is the vector field $\vec{G}(x,y) = \langle xy + y^2, x^2 + 2xy \rangle$ conservative? If so, find a potential function $g(x,y)$ for $\vec{G}(x,y)$.
3. Show that $\vec{F}(x,y) = \langle 3 + 2xy^2, 2x^2y \rangle$ is conservative. Find a potential function $f(x,y)$ for $\vec{F}(x,y)$. Finally, evaluate

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r}$$

where C is the arc of the hyperbola $y = \frac{1}{x}$ from $(1,1)$ to $\left(4, \frac{1}{4}\right)$.

Solutions

1. $\vec{F}(x,y)$ is conservative. A potential function for $\vec{F}(x,y)$ is $f(x,y) = xy^2 - x^2$.
2. $\vec{G}(x,y)$ is not conservative.
3. $\vec{F}(x,y)$ is conservative because, for example:

$$\frac{\partial}{\partial y}(3 + 2xy^2) = 4xy = \frac{\partial}{\partial x}(2x^2y)$$

A potential function for $\vec{F}(x,y)$ is $f(x,y) := 3x + x^2y^2$. Finally, by the Fundamental Theorem for Line Integrals we have:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = f(4, 1/4) - f(1,1) = \boxed{9}$$