16.1: Vector Fields

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Chapter Overview

This chapter breaks completely new ground for us, and begins the basic study of vector calculus, i.e., the calculus of vector fields. A **vector field** $\vec{F}(x, y)$ or $\vec{F}(x, y, z)$ is a function that assigns a vector to every point (x, y) or (x, y, z) in space. We will learn a bit about such functions in the first section of this chapter, and then immediately move into the calculus: line and surface integrals. We will wrap up the course by discussing the connections between these new integrals and the integrals we have already met in this and other courses.

Section Overview

In this section we will define and look at examples of so-called *vector fields*, which will be the chief subject of study for us in this chapter. A point worth noting: I generally shy away from talking too much about applications in this course, as my specialty is pure mathematics. However, the applications of vector fields to science and engineering are numerous! I will do my best to introduce some of these applications where I can, but you should also consult your text (and other courses) for further elaboration and elucidation.

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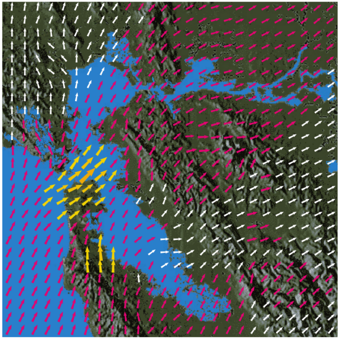
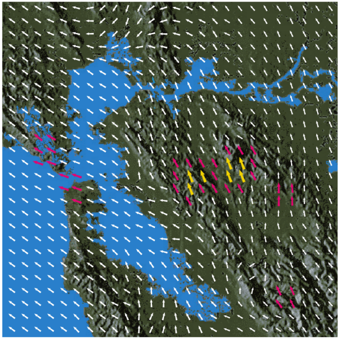
Definition

A **vector field on** $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) is a function $\vec{F}$ that assigns a vector $\vec{F}(x, y)$ (resp. $\vec{F}(x, y, z)$) to each point in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$).

Examples

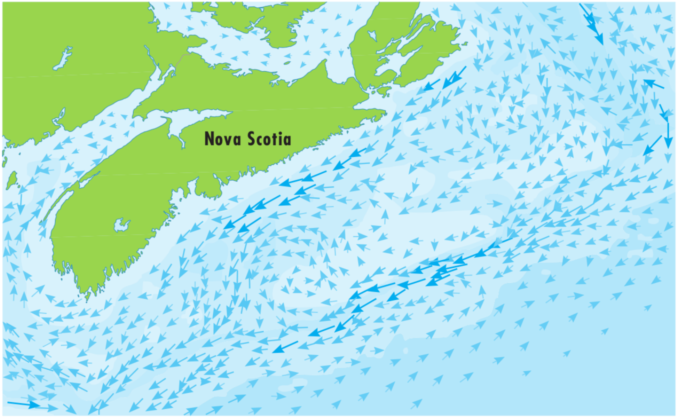
Consider, for example, the following vector fields $\overrightarrow{V}_1(x, y)$ and $\overrightarrow{V}_2(x, y)$, which assign a wind velocity to every point at a given altitude on this two-dimensional map:





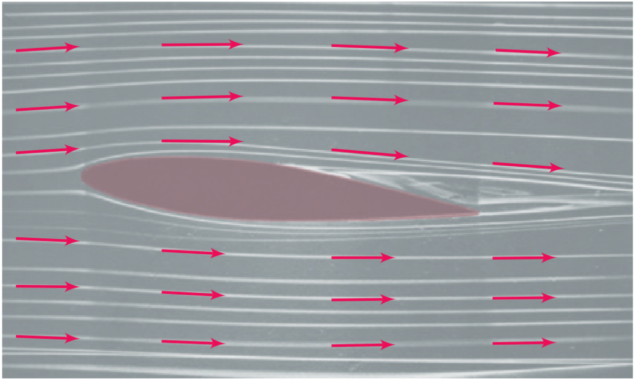
Examples, cont.

Or, consider the vector field $\vec{C}(x, y)$, which assigns a current velocity to every point at a given depth off the coast of Nova Scotia:



Examples, cont.

As one last example, consider $\overrightarrow{W}(x, y)$, which assigns a wind velocity to every point around a two-dimensional slice of this airfoil:



Component Functions

Of course, any vector $\vec{v}$ may be written in component form $\vec{v} = \langle a, b\rangle$ (resp. $\vec{v} = \langle a, b, c\rangle$).

Thus, we define the **component functions** $P(x, y)$ and $Q(x, y)$ (resp. $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$) to be the components of $\vec{F}$ at each point (x, y) (resp. (x, y, z)). That is:

$$\vec{F}(x, y) = \big\langle P(x, y), Q(x, y)\big\rangle$$

(resp. $\vec{F}(x, y, z) = \big\langle P(x, y, z), Q(x, y, z), R(x, y, z)\big\rangle$).

Example

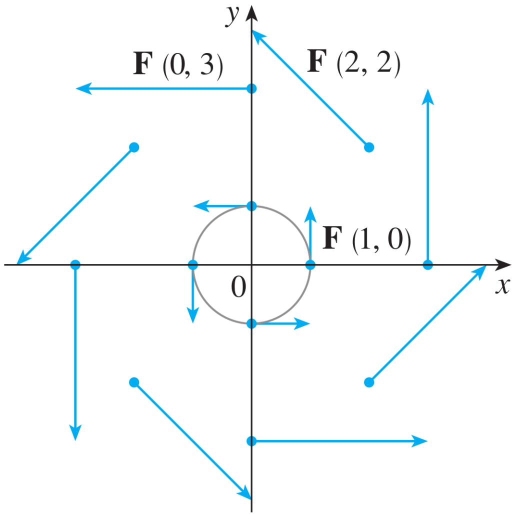
Consider, for example, the vector field $\vec{F}(x,y) = \langle -y, x \rangle$. Let's sketch this vector field.

Let's just pick some points (x,y), and evaluate $\vec{F}(x,y)$ at each of them:

(x,y)	$\vec{F}(x,y)$
$(0,0)$	$\langle 0,0 \rangle$
$(1,0)$	$\langle 0,1 \rangle$
$(2,2)$	$\langle -2,2 \rangle$
$(0,3)$	$\langle -3,0 \rangle$

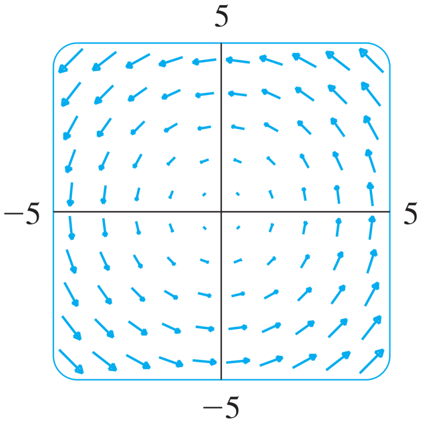
Example, cont.

Continuing in this way, we get an idea of what $\vec{F}(x, y)$ looks like:



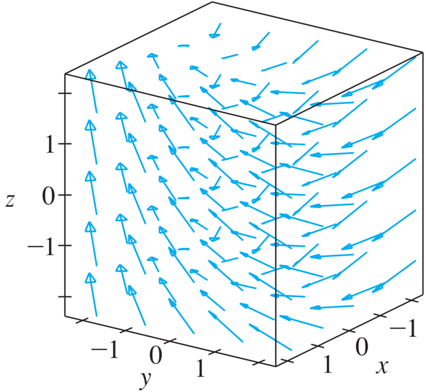
Computers

As you can see, this is an incredibly labor-intensive process. Computers can be a big help in this regard, because they can quickly plot lots of vectors. For example, here's a computer-generated sketch of $\vec{F}(x, y)$ from the previous example:



Computers, cont.

Or, consider the vector field $\vec{F}(x, y, z) = \langle y, -2, x \rangle$:



You can imagine the nightmare that sketching such a vector field by hand would be.

Computers, cont.

In fact, Wolfram alpha will plot vector fields! For example, to plot the field $\vec{F}(x, y) = \langle -y, x \rangle$, you may input the following:

```
plot F(x,y) = <-y,x>
```

The Gradient Vector Field

Believe it or not, there are many deep connections between vector fields and the other topics we've studied in this course. Indeed, much of our work in this chapter will amount to elucidating the ways that the calculus of vector fields connects to the calculus of both curves and surfaces.

Even at this early stage, we can begin to see a sliver of these connections.

Recall that the gradient of a two-variable function $f(x, y)$ is:

$$\nabla f(x, y) = \left\langle f_x(x, y), f_y(x, y) \right\rangle$$

In particular, the gradient of $f(x, y)$ is a vector field. Imaginatively, we refer to this as the **gradient vector field** of $f(x, y)$. (We can also define the gradient vector field for functions of more variables similarly).

An Observation

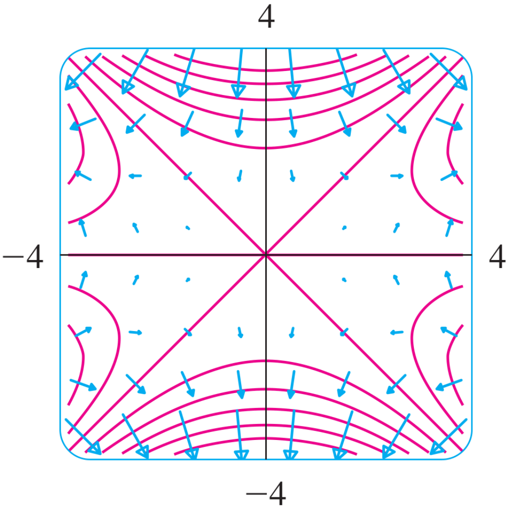
Plot the contour map of $f(x, y) = x^2y - y^3$ and its gradient vector field on the same axes, and see how they are related.

First, the gradient vector field of $f(x, y)$ is

$$\nabla f(x, y) = \left\langle 2xy, x^2 - 3y^2 \right\rangle$$

On the following slide, we graph some of the vectors in this vector field (blue) and some contour lines of $f(x, y)$ (pink) on the same plot:

An Observation, cont.



An Observation, cont.

Note that the vectors are perpendicular to the contour lines (which is something we talked a bit about in chapter 14)!

So, if we have a vector field $\vec{F}(x, y)$ and we can find a function $f(x, y)$ such that $\nabla f(x, y) = \vec{F}(x, y)$, then a sketch of some level curves of $f(x, y)$ can help us to sketch $\vec{F}(x, y)$.

By the way, a vector field $\vec{F}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ with $\nabla f = \vec{F}$ for some function f is called a **conservative vector field**. f is called a **potential function** for $\vec{F}$. Many conservative vector fields arise in applications, and we will have cause to discuss them extensively in this chapter.

Notation and Definition

We will often wish to make general statements that are true of vector fields on $\mathbb{R}^n$ for n an arbitrary integer. To help make writing such statements more efficient, we will often conflate tuples (x, y), (x, y, z), etc. with the vectors $\langle x, y\rangle$, $\langle x, y, z\rangle$, etc., respectively, and write the generic expression $\vec{F}(\vec{x})$ as a stand-in for $\vec{F}(x, y)$ or $\vec{F}(x, y, z)$, etc.

Here's a definition, stated with this new notation:

We call a vector field $\vec{F}(\vec{x})$ **continuous** if each of its component functions is continuous.

(In particular, this definition is given for $\vec{x}$ a vector (but really, an n-tuple) in $\mathbb{R}^n$ for any integer value of n).

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Exercises

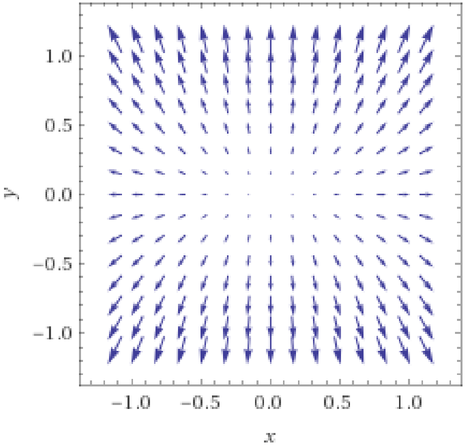
1. Sketch the vector field $\overrightarrow{G}_1(x, y) = \left\langle \frac{x}{2}, y \right\rangle$.
2. Sketch the vector field $\overrightarrow{G}_2(x, y, z) = \langle 1, 0, 1 \rangle$.
3. Find the gradient vector field $\nabla g_3(x, y)$ for

$$g_3(x, y) = \frac{1}{2}(x^2 - y^2)$$

   Then, use level curves of $g_3(x, y)$ to help sketch this gradient vector field.

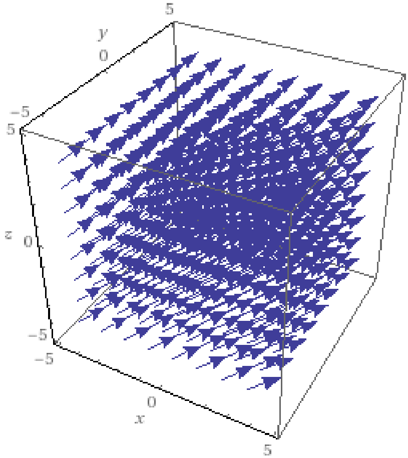
Solutions

1. Here's a sketch of $\overrightarrow{G}_1(x, y)$:



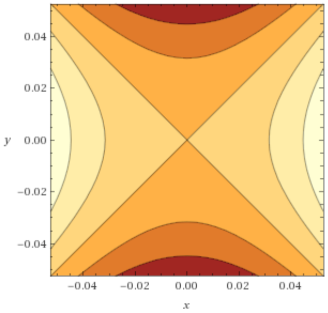
Solutions, cont.

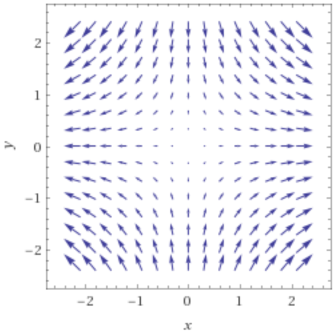
2. Here's a sketch of $\vec{G}_2(x, y)$:



Solutions, cont.

3. The gradient vector field of $g_3(x, y)$ is $\nabla g_3(x, y) = \langle x, -y \rangle$. The contour plot of $g_3(x, y)$ and the gradient vector field $\nabla g_3(x, y)$ are:





Note that the vectors in the latter are perpendicular to the contour lines in the former!