# 15.8: Triple Integrals in Spherical Coordinates

Julia Jackson

Department of Mathematics
The University of Oklahoma

Fall 2021

## Overview

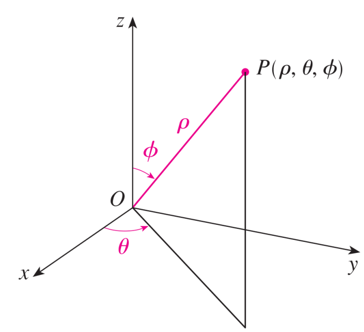
In the previous section we learned about cylindrical coordinates, which can be used, albeit somewhat indirectly, to help us efficiently evaluate triple integrals of three-variable functions over type 1 subsets of their domains whose projections onto the *xy*-plane may be parametrized with polar coordinates.

In sharp contrast to the previous section, in this one we will learn about spherical coordinates. These will be used to aid us in evaluating triple integrals of three-variable functions over subsets of their domains that are bounded by cones and spheres, in a manner that is generally much more efficient than using rectangular or cylindrical coordinates. We will also use this coordinate system directly, i.e. we will absolutely be aware that we are using spherical coordinates to compute these triple integrals, so mastery of this coordinate system will be indispensable.

## Table of Contents



## Spherical Coordinates

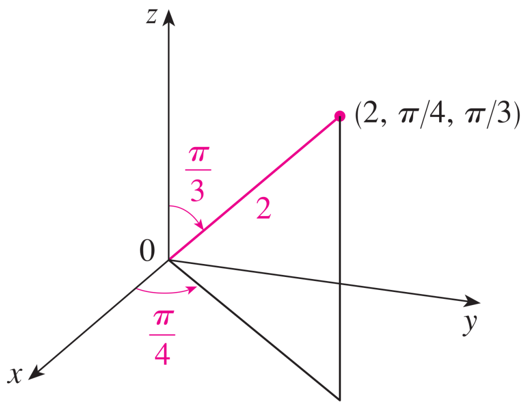



The **spherical coordinates** $(\rho, \theta, \phi)$ of a point *P* are as follows: $\rho$ is the (non-negative) distance from the origin *O* to *P*, and $\phi$ is the angle between 0 and $\pi$ that is formed by the positive *z*-axis and the line segment *OP*. Finally, if we project *OP* onto the *xy*-plane, $\theta$ is the angle that this projection makes with the positive *x*-axis, i.e., the same $\theta$ we had in cylindrical coordinates.

## Example

Plot the point $P = (2, \pi/4, \pi/3)$, which is given in spherical coordinates.

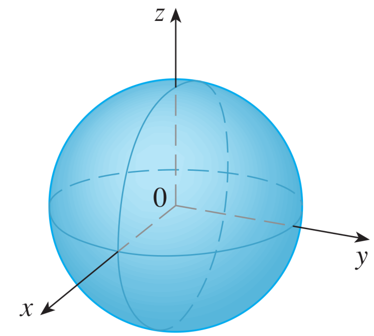
We have:




## Example

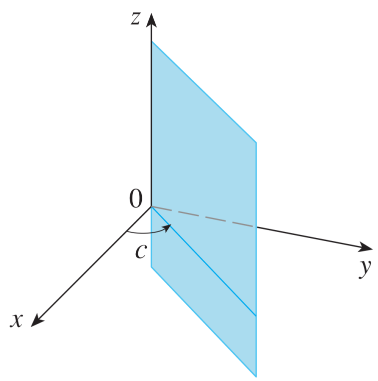
Let's get a better handle on things by graphing some basic functions given in spherical coordinates. Let $c$ be a constant. Sketch the surfaces $\rho = c$, $\theta = c$, and $\phi = c$.

First, consider $\rho = c$. This is the set of all points in $\mathbb{R}^3$ whose distance from the origin is $c$, i.e., a sphere of radius $c$:




## Example, cont.

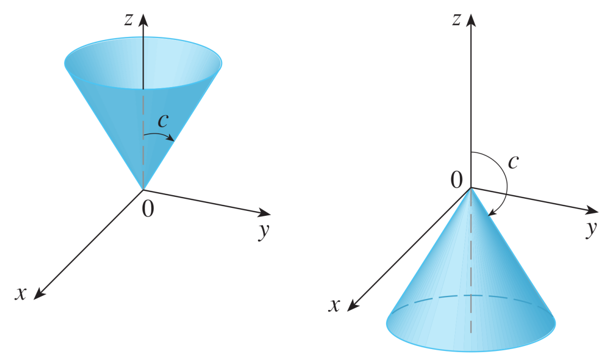
Next, consider $\theta = c$. This is the set of all points in $\mathbb{R}^3$ whose projections onto the $xy$-plane lie on the ray extending from the origin which makes an angle of $c$ with the positive $x$-axis, as follows:




In other words, this is half of a plane.

## Example, cont.

Finally, let's consider $\phi = c$. This is the set of all points in $\mathbb{R}^3$ which lie on any ray extending from the origin that makes an angle of $c$ with the positive $z$-axis, as follows:




$0 < c < \pi/2$　　　$\pi/2 < c < \pi$

In other words, this is a cone (well... most of the time, anyway. What happens when $\phi$ is 0, $\frac{\pi}{2}$, or $\pi$?).

# Conversion

When we come to using spherical coordinates to evaluate triple integrals, we will regularly need to convert from rectangular to spherical coordinates. We give the most common conversions that we will use for this task here.

Let a point $P$ have spherical coordinates $(\rho, \theta, \phi)$ and rectangular coordinates $(x, y, z)$. One crucial relationship between these coordinates is also fairly straightforward to see:

$$\rho^2 = x^2 + y^2 + z^2$$

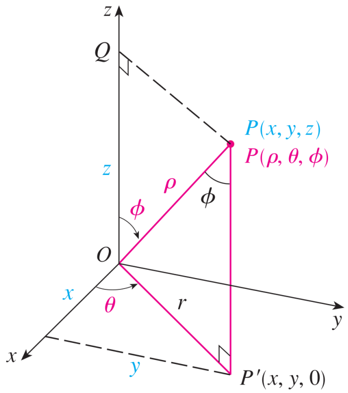
Indeed, $\rho$ is the distance from the origin to the point $P$. We can also show that we have the relationships:

$$x = \rho \sin(\phi)\cos(\theta) \quad y = \rho \sin(\phi)\sin(\theta) \quad z = \rho\cos(\phi)$$

Let's show one of these; I'll leave the others as an exercise.

# Conversion, cont.

We have the following handy picture:




From this picture, we can see that $x = r\cos(\theta)$, and furthermore that $r = \rho\sin(\phi)$. Therefore, combining we have $x = \rho\sin(\phi)\cos(\theta)$, as claimed.

# Example

Let $P = (2, \pi/4, \pi/3)$ be given in spherical coordinates. Find its rectangular coordinates.
Recall that:

$$x = \rho \sin(\phi)\cos(\theta) \quad y = \rho \sin(\phi)\sin(\theta) \quad z = \rho\cos(\phi)$$

Therefore, we have:

$$x = 2\sin(\pi/3)\cos(\pi/4) = 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \sqrt{\frac{3}{2}}$$

$$y = 2\sin(\pi/3)\sin(\pi/4) = 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \sqrt{\frac{3}{2}}$$

$$z = \rho\cos(\phi) = 2\cos(\pi/3) = 2\cdot\frac{1}{2} = 1$$

So, $\boxed{P = \left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)}$ in rectangular coordinates.

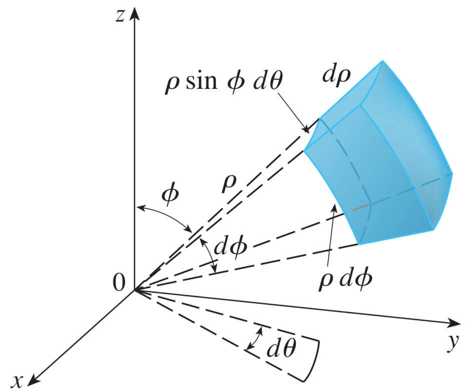
# Table of Contents

## Spherical Wedges

Suppose now that we want to integrate a function $f(x, y, z)$ over a so-called **spherical wedge** $E$:

$$E = \left\{ (\rho, \theta, \phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d \right\}$$

This looks something like the following (you may ignore the labels):




## The Formula

To integrate over such a region $E$, we use the following formula:

$$\iiint_E f(x, y, z)\, \mathrm{d}V$$
$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin(\phi)\cos(\theta), \rho \sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2 \sin(\phi)\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\phi$$

Where, again,

$$E = \left\{ (\rho, \theta, \phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d \right\}$$

with $a \ge 0$, $\beta - \alpha \le 2\pi$ and $d - c \le \pi$. There is a strong analogy to double integrals in polar coordinates here. We convert the integrand from rectangular to spherical coordinates, and then (for technical reasons) multiply by an extra term of $\rho^2 \sin(\phi)$ (analogous to the extra $r$ in polar coordinates).

## Example

Evaluate

$$I_1 = \iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, \mathrm{d}V$$

where $B$ is the unit ball

$$B = \left\{ (x, y, z) \mid x^2 + y^2 + z^2 \le 1 \right\}$$

First, note that we may describe the unit ball as follows:

$$B = \left\{ (\rho, \theta, \phi) \mid 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi \right\}$$

## Example, cont.

Therefore, we have:

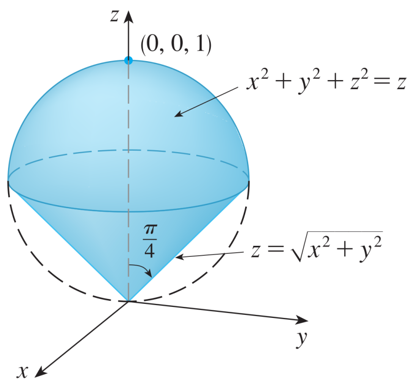
$$I_1 = \int_0^\pi \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin(\phi)\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\phi$$
$$= \int_0^\pi \int_0^{2\pi} \int_0^1 \rho^2 e^{\rho^3} \sin(\phi)\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\phi$$
$$= \boxed{\frac{4}{3}\pi(e-1)}$$

where we complete the iterated integral using the substitution $u = \rho^3$.

# Example

Calculate the volume $V_2$ of the solid $E$ that lies above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = z$.

Let's begin by sketching $E$, using the techniques for sketching surfaces that we've learned previously:




# Example, cont.

Next, recall from section 15.6 that the volume $V_2$ of $E$ is given by the integral:

$$V_2 = \iiint_E 1 \, dV$$

Since the boundaries on the solid $E$ are a sphere and a cone, spherical coordinates are an excellent coordinate system to try and use to evaulate this integral. Thus, our next goal is to parametrize $E$ in spherical coordinates.

We may begin as follows:

$$E = \left\{ (\rho, \theta, \phi) \;\middle|\; 0 \leq \phi \leq \pi/4, 0 \leq \theta \leq 2\pi, 0 \leq \rho \leq \boxed{?} \right\}$$

# Example, cont.

Why is this description accurate, up to this point? Well, to discover the bounds on $\rho$, $\theta$, and $\phi$ which collect all of the points in a given region, it's nearly always best to take these in reverse order.

First, we need to figure out the smallest and largest values of $\phi$ which correspond to points inside of our region, where, of course, these angles will always be somewhere between 0 and $\pi$. By inspection, we see that the cone $z = \sqrt{x^2 + y^2}$ makes an angle of $\frac{\pi}{4}$ with the positive $z$-axis (use the traces $x = 0$ and/or $y = 0$ to help see this), so that the smallest and largest values of $\phi$ which correspond to points inside our region are 0 and $\frac{\pi}{4}$.

# Example, cont.

Next, if we fix a value of $\phi$ between 0 and $\frac{\pi}{4}$, we see that there are points inside of $E$ at every value of $\theta$ between 0 and $2\pi$; indeed, at such a $\phi$, there are points inside of $E$ in a complete circle around the $z$-axis.

Finally, fix a $\phi$ between 0 and $\frac{\pi}{4}$ and a $\theta$ between 0 and $2\pi$. The points in $\mathbb{R}^3$ with these two spherical coordinates form a ray extending from the origin. Let's call the ray for our chosen $\phi$ and $\theta$ $R$. We see that $R$ intersects $E$ both at the origin and then once again at a point which we will call $P_R$ on the surface $x^2 + y^2 + z^2 = z$. Therefore, the points on $R$ which are inside of $E$ have a $\rho$-coordinate between 0 and... whatever the $\rho$-coordinate of $P_R$ is.

## Example, cont.

To help us figure out what the $\rho$-coordinate of $P_R$ is, we begin by writing the equation $x^2 + y^2 + z^2 = z$ in spherical coordinates. Our coordinate conversion formulas give:

$$\begin{aligned} x^2 + y^2 + z^2 = z &\Rightarrow \rho^2 = \rho\cos(\phi) \\ &\Rightarrow \rho = \cos(\phi) \end{aligned}$$

Therefore, at the $\rho$-coordinate of $P_R$ is $\cos(\phi)$ for our chosen value of $\phi$, and therefore we have that:

$$E = \left\{ (\rho, \theta, \phi) \;\middle|\; 0 \leq \phi \leq \pi/4, 0 \leq \theta \leq 2\pi, 0 \leq \rho \leq \cos(\phi) \right\}$$

## Example, cont.

Therefore, we have:

$$\begin{aligned} V_2 &= \iiint_E 1 \, \mathrm{d}V \\ &= \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\cos(\phi)} \rho^2 \sin(\phi) \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \boxed{\frac{\pi}{8}} \end{aligned}$$

(Verify this calculation)

## Table of Contents



## Exercises

1. Use the diagram in the slides to demonstrate the relationships $y = \rho\sin(\phi)\sin(\theta)$ and $z = \rho\cos(\phi)$ that help convert between rectangular and spherical coordinates.
2. Plot the points $P = (2, \pi/2, \pi/2)$ and $Q = (4, -\pi/4, \pi/3)$ given in spherical coordinates. Then give the rectangular coordinates of these points.
3. Evaluate $I_3 = \iiint_B (x^2 + y^2 + z^2)^2 \, \mathrm{d}V$ where $B$ is the ball centered at the origin of radius 5.
4. Evaluate $I_4 = \iiint_E y^2 z^2 \, \mathrm{d}V$, where $E$ is the region in $\mathbb{R}^3$ which lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 1$.

# Solutions

1. If you'd like any help working this out, please get in touch!
2. In rectangular coordinates, $P = (0, 2, 0)$ and $Q = (\sqrt{6}, -\sqrt{6}, 2)$.
3. $$I_3 = \int_0^{\pi} \int_0^{2\pi} \int_0^{5} \rho^6 \sin(\phi) \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi = \boxed{\frac{312500\pi}{7}}$$
4. $$I_4 = \int_0^{\pi/3} \int_0^{2\pi} \int_0^{1} \rho^6 \sin^3(\phi) \cos^2(\phi) \sin^2(\theta) \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi = \boxed{\frac{47\pi}{3360}}$$