15.6: Triple Integrals

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Overview

Up to this point in the chapter, we have only discussed double integrals, i.e. the integrals of two-variable functions $f(x, y)$. These allow us to find the signed volume in $\mathbb{R}^3$ of the solid between a surface $z = f(x, y)$ and a region R in the xy-plane. We learned to evaluate such integrals using the technique of *iterated integrals*.

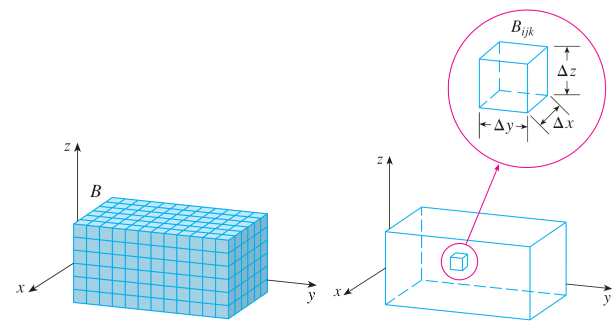
In the next three sections we turn our attention to triple integrals, i.e. integrals of functions of three variables $f(x, y, z)$. There's no simple graphical interpretation for such integrals, but the core concept translates perfectly well and has useful applications.

With some effort, one could define the integral of a function of arbitrarily-many variables, but in general this is a very complicated proposition. I invite you to consider why this is the case as we proceed.

Table of Contents

The Setup

Suppose that we have a function $f(x, y, z)$ of three variables which is continuous on a rectangular box B. We may formally extend the method of integration we learned for one- and two-variable functions to integrating f over B in the expected way: First split B into sub-boxes B_{ijk}, and choose a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box:



The Setup, cont.

Now, let ΔV be the volume of each sub-box B_{ijk}. Evaluate f at each sample point and multiply each result by ΔV. Add up the products to form a Riemann sum:

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

This approximates the four-dimensional signed "volume" between the "graph" of f and the region B, which we call the triple integral of f over B. To find the actual volume, we take a limit, as usual:

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

The Graphical Interpretation

Intuitively, the triple integral gives the volume of the four-dimensional solid between the rectangular box B and the graph of f. But note that we can draw neither the solid nor the graph of a function $w = f(x,y,z)$, as both would require a four-dimensional drawing.

So, I'll say this one last time: there is no nice graphical interpretation for the triple integral of a three-variable function $f(x,y,z)$. We have merely imitated the argument for lower-dimensional integrals to devise this new concept.

However, that does not mean that such integrals are not useful, nor does it mean that we cannot calculate them.

A Physical Interpretation

Indeed, here's one application of the triple integral of a three-variable function:

Suppose that the function $f(x,y,z)$ above is a density function of the box B. Then the triple integral

$$\iiint_B f(x,y,z)\,\mathrm{d}V$$

gives the mass of B; I'll leave it to you to run through the argument we used to build the triple integral to see why this is so.

Your text also contains other applications, and I encourage you to take a look at these. For the most part, we will talk about the abstract mathematical process of evaluating these integrals more than how they are used in practice — the latter will vary, depending on your field of study.

Calculation

We certainly wouldn't want to evaluate $\iiint_B f(x,y,z)\,\mathrm{d}V$ using the limit definition from above; one would hope there is an easier way. And there is! We can calculate the triple integral as an iterated integral:

Theorem (Fubini): If f is continuous on the rectangular box $B = [a,b]\times[c,d]\times[r,s]$, then we may calculate $\iiint_B f(x,y,z)\,\mathrm{d}V$ as an iterated integral:

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \int_r^s\int_c^d\int_a^b f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Furthermore, any of the six possible orders of integration will yield the same result.

Example

Evaluate

$$I_1 = \iiint_B xyz^2 \, \mathrm{d}V$$

where B is the rectangular box $B = [0,1] \times [-1,2] \times [0,3]$.

By Fubini's theorem we may express I_1 as an iterated integral in six distinct ways, and we are free to choose an order of integration that suits this problem. Since the integrand itself presents no obvious order of integration to try, we are free to set up whatever order of integration we like. Here's the order I've chosen: I would rank the bounds on variables from friendliest to least friendly this way: x, then z, then y. Therefore, this is the order I will choose to integrate in.

Example, cont.

We have:

$$\begin{aligned} I_1 &= \int_{-1}^{2} \int_0^3 \int_0^1 xyz^2 \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}y \\ &= \int_{-1}^{2} \int_0^3 \left(\frac{x^2}{2} yz^2 \right) \Bigg|_{x=0}^{x=1} \mathrm{d}z \, \mathrm{d}y \\ &= \int_{-1}^{2} \int_0^3 \frac{1}{2} yz^2 \, \mathrm{d}z \, \mathrm{d}y \\ &= \int_{-1}^{2} \frac{9}{2} y \, \mathrm{d}y \\ &= \frac{9}{4} y^2 \Big|_{-1}^{2} = \boxed{\frac{27}{4}} \end{aligned}$$

Table of Contents

Type 1 Regions

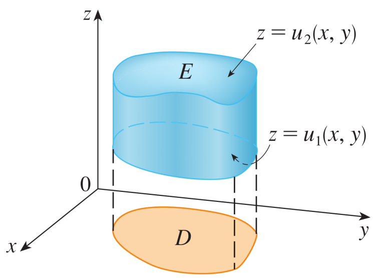
Of course, there's no reason that we should restrict ourselves to integrating over boxes. There are lots of other bounded solids E in R^3 that we could integrate over. We will investigate some of these now.

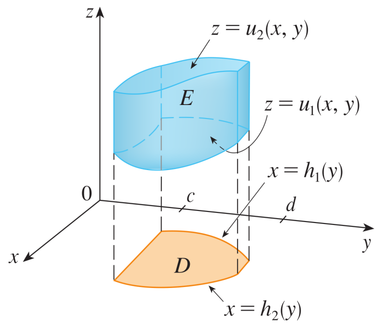
The first we will encounter are type 1 regions. E is said to be of **type 1** if it lies above and/or below a region D in the xy-plane and between two continuous functions of x and y. That is, if we may describe E as follows:

$$E = \big\{(x, y, z) | (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\big\}$$

Type 1 Regions, cont.

Two examples:





Integrating Over a Type 1 Region

If E is a type one region as above, then we may evaluate $\iiint_E f(x,y,z)\,\mathrm{d}V$ as follows:

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,\mathrm{d}z\right] \mathrm{d}A$$

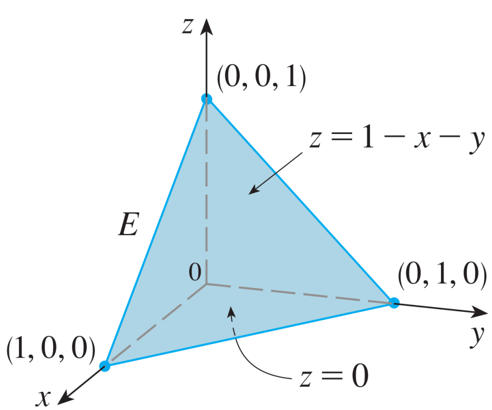
Where D is the base region in the xy-plane beneath E (i.e. the projection of E onto the xy-plane).

After setting this step up, we set up the double integral over D exactly as we learned previously.

Example

Evaluate $I_2 = \iiint_E z\,\mathrm{d}V$ where E is the tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

We begin by drawing E:



Example, cont.

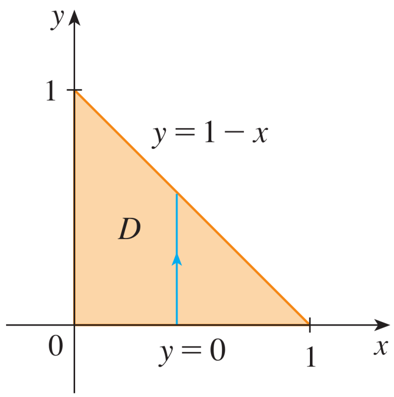
Now, note that E is trapped above and below by the functions $z = 1 - x - y$ and $z = 0$, respectively. Therefore, from above we have:

$$\begin{aligned} I_2 &= \iint_D \left[\int_0^{1-x-y} z\,\mathrm{d}z\right] \mathrm{d}A \\ &= \iint_D \frac{1}{2}z^2\Big|_0^{1-x-y} \mathrm{d}A \\ &= \iint_D \frac{1}{2}(1-x-y)^2\,\mathrm{d}A \end{aligned}$$

where D is the projection of E onto the xy-plane.

Example, cont.

Next, let's sketch and parametrize D. We have:



Example, cont.

We can think of this region as either type I or type II. The double integral above doesn't point us in one direction or another, so we'll just think of it as type 1, as follows:

$$D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$$

Example, cont.

Thus, we have:

$$\begin{aligned} I_2 &= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)^2 \, dy \, dx \\ &= \int_0^1 \frac{-1}{6}(1-x-y)^3 \Big|_0^{1-x} dx \\ &= \int_0^1 \frac{1}{6}(1-x)^3 \, dx \\ &= \frac{-1}{24}(1-x)^4 \Big|_0^1 \\ &= \boxed{\frac{1}{24}} \end{aligned}$$

Type 2 Regions

The second type of region we might integrate over is a type 2 region. A region E is said to be of **type 2** if it lies in front of and/or behind a region D in the yz-plane and between two continuous functions of y and z. That is, if we may describe E as follows:

$$E = \{(x, y, z) | (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

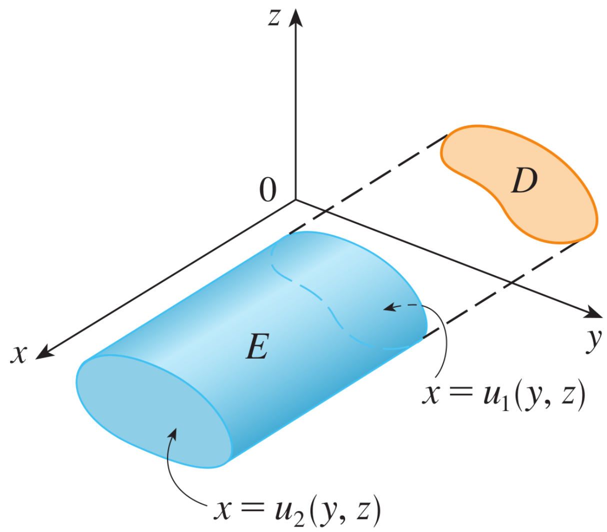
If E is a type 2 region, we evaluate $\iiint_E f(x, y, z) \, dV$ as follows:

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z) \, dx \right] dA$$

analogously to how we evaluated integrals over type 1 regions.

A Type 2 Region

Here's an example of a type 2 region:



Type 3 Regions

The final type of region we might integrate over is a type 3 region. A region E is said to be of **type 3** if it lies to the right and/or left of a region D in the xz-plane and between two continuous functions of x and z. That is, if we may describe E as follows:

$$E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

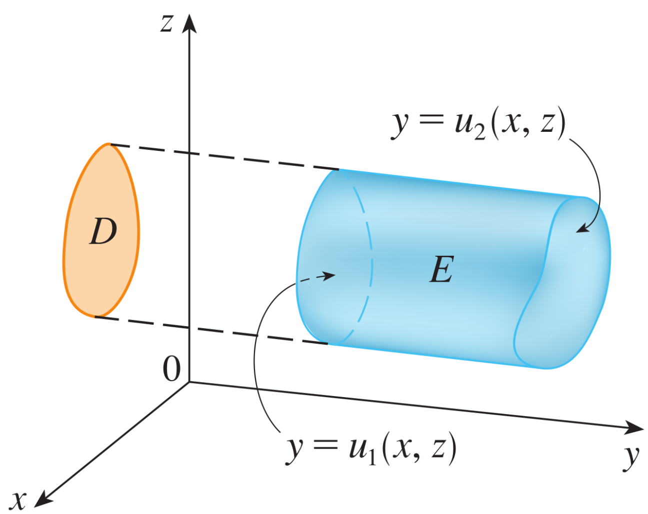
If E is a type 2 region, we evaluate $\iiint_E f(x, y, z)\,dV$ as follows:

$$\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\,dy \right] dA$$

analogously to how we evaluated integrals over type 1 and 2 regions.

A Type 2 Region

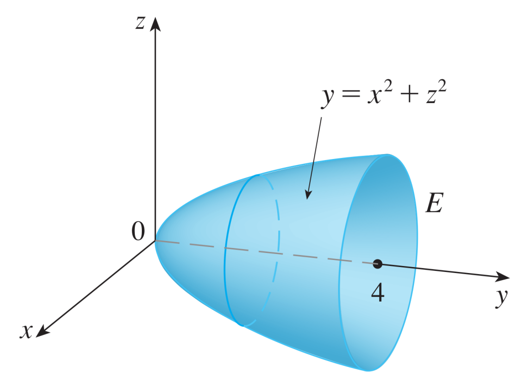
Here's an example of a type 3 region:



Example

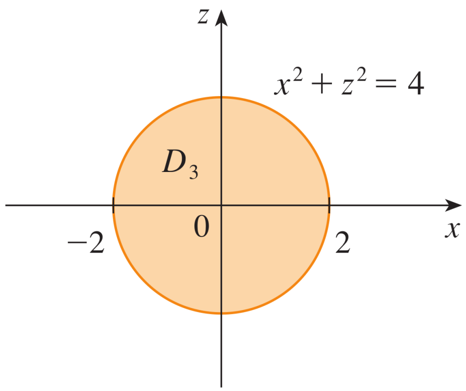
Evaluate $I_3 = \iiint_E \sqrt{x^2 + z^2}\,dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Let us begin by sketching E:



Example, cont.

You could conceive of E as a type 1, type 2, or type 3 region, but I think it is easiest to think of it as the latter. Indeed, E lies neatly between the function $y = x^2 + z^2$ and the plane $y = 4$, and its projection onto the xz-plane is the following:



Example, cont.

Therefore, we have:

$$I_3 = \iint_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, \mathrm{d}y \right] \mathrm{d}A$$
$$= \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2+z^2}\, \mathrm{d}A$$

Example, cont.

Now, D_3 can be neatly parametrized as a polar region in the xz-plane:

$$D_3 = \left\{ (r, \theta) | 0 \le \theta \le 2\pi, 0 \le r \le 2 \right\}$$

Furthermore, since D_3 lies in the xz-plane, we have the polar relations $x^2 + z^2 = r^2$, $x = r\cos(\theta)$, and $z = r\sin(\theta)$.

Thus:

$$I_3 = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2+z^2}\, \mathrm{d}A$$
$$= \int_0^{2\pi} \int_0^2 (4 - r^2)\sqrt{r^2} r\, \mathrm{d}r\, \mathrm{d}\theta$$
$$= \int_0^{2\pi} \int_0^2 (4r^2 - r^4)\, \mathrm{d}r\, \mathrm{d}\theta = \boxed{\frac{128\pi}{15}}$$

Two Final Facts

Of course, most regions in $\mathbb{R}^3$ are not type 1, type 2, or type 3. So, just as with double integrals, to evaluate $\iiint_E f(x, y, z)\, \mathrm{d}V$ over such a region E, we break E into subregions $E_1, E_2, \ldots, E_n$ which are each one of these types, and sum the integrals over these instead:

$$\iiint_E f(x,y,z)\, \mathrm{d}V = \iiint_{E_1} f(x,y,z)\, \mathrm{d}V + \cdots + \iiint_{E_n} f(x,y,z)\, \mathrm{d}V$$

Two Final Facts, cont.

We previously stated that the area $A(R)$ of a region R in $\mathbb{R}^2$ could be calculated with a double integral:

$$A(R) = \iint_R 1 \, \mathrm{d}A$$

Similarly, the volume $V(E)$ of a region E in $\mathbb{R}^3$ may be calculated with a triple integral:

$$V(E) = \iiint_E 1 \, \mathrm{d}V$$

Table of Contents

Exercises

1. Calculate the volume V_1 of the rectangular box $B = [0,1] \times [-1,2] \times [0,3]$ in two ways: first, directly using the formula for the volume of a rectangular box; and, second, as a triple integral.
2. Calculate the volume V_2 of the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.
3. Evaluate $I_3 = \iiint_T x^2 z \, \mathrm{d}V$ where T is the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.
4. Rewrite the iterated integral

$$I_4 = \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$$

as an iterated integral in a different order, integrating first with respect to x, then z, then y [Hint: begin by sketching the region E of integration using the bounds of the iterated integral].

Solutions

1. Either method should yield $V_1 = 9$.
2. One possible solution: $V_2 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{6}$
3. One possible solution: $I_3 = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-2y-x} x^2 z \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{90}$
4. $I_4 = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z) \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}y$