15.5: Surface Area

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Fall 2021

Overview

We now have two complementary techniques we can use to evaluate a double integral of a (continuous) two-variable function over a general subset of its domain in the xy-plane, so it's probably about time that we come to an application.

You may remember from single-variable calculus that the length of an arc on the graph of a single-variable function can be calculated using an (ordinary) integral. In this section, we use double integrals to solve the analogous problem for two-variable functions: calculating the amount of area of a piece of the graph of a two-variable function (i.e. a surface area) in $\mathbb{R}^3$.

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Review: Arc Length

Suppose that we have a single-variable function $f(x)$ which is continuous on the interval $[a, b]$. If we wish to measure the "size" L of the portion of the graph of $f(x)$ that lies above and/or below $[a, b]$ (i.e. the length of the arc on the graph of f on the interval $[a, b]$), we have the very convenient arc length formula:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x$$

Two-Variable Functions

Now consider the analogous question for a two-variable function $f(x, y)$: Suppose that $f(x, y)$ is continuous on the region D in the xy-plane, and that we wish to measure the "size" $A(S)$ of the portion S of the graph of $f(x, y)$ that lies above and/or below D. The graph of $f(x, y)$ is, of course, a surface, so what we want is to calculate the area of S. It turns out that this area is given by:

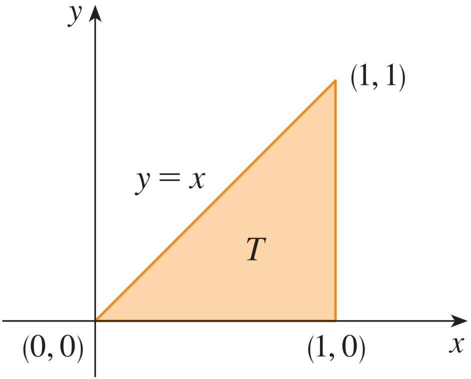
$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, \mathrm{d}A$$

Note that this formula, too, is analogous to the formula from the single-variable case.

Example

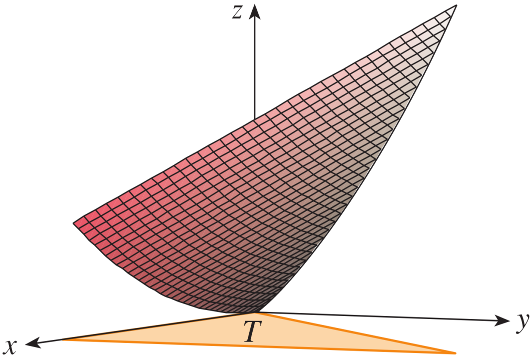
Find the surface area A of the portion of the surface $z = x^2 + 2y$ that lies above the region T in the xy-plane bounded by the triangle with vertices $(0, 0), (1, 0)$, and $(1, 1)$.

Let's begin by drawing the region T:



Example, cont.

Here is a picture of the portion of the surface $z = x^2 + 2y$ lying above T, purely for your reference (you don't need to provide such a sketch for a complete solution):



We wish to find the area of this portion of the surface.

Example, cont.

Let $f(x, y) = x^2 + 2y$. Then the graph of $f(x, y)$ is $z = x^2 + 2y$, and the surface area formula gives:

$$\begin{aligned} A &= \iint_T \sqrt{[2x]^2 + [2]^2 + 1}\,\mathrm{d}A \\ &= \iint_T \sqrt{4x^2 + 5}\,\mathrm{d}A \\ &= \int_0^1 \int_0^x \sqrt{4x^2 + 5}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 x\sqrt{4x^2 + 5}\,\mathrm{d}x \end{aligned}$$

Let $u = 4x^2 + 5$ so that $\frac{1}{8}\,\mathrm{d}u = x\,\mathrm{d}x$, giving:

Example, cont.

$$
\begin{aligned}
A &= \int_5^9 \frac{1}{8}\sqrt{u}\,\mathrm{d}u \\
&= \frac{1}{12}u^{3/2}\Big|_5^9 \\
&= \boxed{\frac{1}{12}(27 - 5\sqrt{5})}
\end{aligned}
$$

Example

Find the area B of the paraboloid $z = x^2 + y^2$ that lies above the disk D in the plane given by $x^2 + y^2 \leq 9$.

If we let $f(x, y) = x^2 + y^2$, then notice that the graph of $f(x, y)$ is $z = x^2 + y^2$. Note also that D is described most easily in polar coordinates as:

$$D = \left\{(r, \theta) | 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\right\}$$

Example, cont.

Therefore, using the surface area formula from a previous slide we have:

$$
\begin{aligned}
B &= \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\,\mathrm{d}A \\
&= \iint_D \sqrt{4x^2 + 4y^2 + 1}\,\mathrm{d}A \\
&= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2\cos^2(\theta) + 4r^2\sin^2(\theta)}\,r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^3 r\sqrt{1 + 4r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \boxed{\frac{\pi}{6}(37\sqrt{37} - 1)}
\end{aligned}
$$

where the inner partial integral of this iterated integral may be completed via u-substitution.

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Exercises

1. Find the area A_1 of the part of the plane $z = 5x + 3y + 6$ that lies above the rectangle $R_1 = [1, 4] \times [2, 6]$.
2. Find the area A_2 of the part of the plane $6x + 4y + 2z = 1$ that lies inside the cylinder $x^2 + y^2 = 25$.
3. Find the area A_3 of the part of the surface $2x + 4z - y^2 = 5$ that lies above the triangle T_3 with vertices $(0, 0)$, $(0, 2)$, and $(4, 2)$.

Solutions

1. One possible solution is: $A_1 = \int_1^4 \int_2^6 \sqrt{35}\,\mathrm{d}y\,\mathrm{d}x = 12\sqrt{35}$
2. One possible solution is: $A_2 = \int_0^{2\pi} \int_0^5 r\sqrt{14}\,\mathrm{d}r\,\mathrm{d}\theta = 25\pi\sqrt{14}$
3. One possible solution is: $A_3 = \int_0^2 \int_0^{2y} \sqrt{\frac{5}{4} + \frac{1}{4}y^2}\,\mathrm{d}x\,\mathrm{d}y = 9 - \frac{5^{3/2}}{3}$