15.3: Double Integrals in Polar Cooridinates

Julia Jackson

Department of Mathematics
The University of Oklahoma

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Overview

Due to our work over the previous two sections, at this point we could theoretically evaluate the double integral of any continuous two-variable function over any subset D of its domain in the xy-plane.

However, some subsets of the domain of a two-variable function are particularly tricky to work with using only the tools we've developed so far. In this section, we will examine how to calculate double integrals over regions that are most easily described using polar coordinates.

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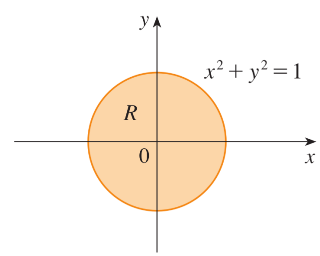
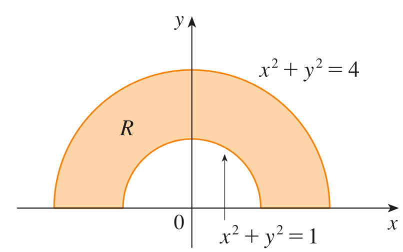
The Setup

Consider the following subets of the xy-plane:





Suppose that we have a two-variable function $f(x, y)$ which is continuous on these regions, and we wish to evaluate the double integral of $f(x, y)$ on either. Let's try setting up such a double integral as an iterated integral using the techniques we've developed up to this point [give this a try before reading on!].

Region 1

The first region on the previous slide, which we will henceforth refer to as R_1 (my apologies for the conflict with the labelling on the previous slide; those images were pulled directly from your text) can be thought of as either of type I or type II, though this may not be obvious at first. If we solve the equation of the bounday for y, we get:

$$y = \pm\sqrt{1 - x^2}$$

Recall that the positive square root is an equation of the top half of the circle, while the negative square root is an equation of the bottom half of the circle. Therefore, we may describe R as:

$$R_1 = \left\{(x, y) \mid -1 \leq x \leq 1,\ -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}\right\}$$

Region 1, cont.

Therefore, with this description in mind, we have:

$$\iint_{R_1} f(x, y)\,\mathrm{d}A = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

Now, imagine that $f(x, y)$ were a particularly nice function; perhaps a polynomial. Imagine computing a partial antiderivative of this $f(x, y)$... and plugging in those square root endpoints. This could be an absolute nightmare! And R_1 is a comparatively simple shape: a disk!

Region 2

The second region above, which we will call R_2 from here on out, is even worse. It isn't strictly of type I or type II, but it can be broken into a minimum of three separate pieces that are each of type I, or into a minimum of three separate pieces that are each of type II. We leave the following description of R_2 as a union of three type I pieces which only overlap on their boundaries as an exercise:

$$\begin{aligned} R_2 = &\left\{(x, y) \mid -2 \leq x \leq -1,\ 0 \leq y \leq \sqrt{4 - x^2}\right\} \\ &\cup \left\{(x, y) \mid -1 \leq x \leq 1,\ \sqrt{1 - x^2} \leq y \leq \sqrt{4 - x^2}\right\} \\ &\cup \left\{(x, y) \mid 1 \leq x \leq 2,\ 0 \leq y \leq \sqrt{4 - x^2}\right\} \end{aligned}$$

Region 2, cont.

Therefore, by our work in the previous section, we see that:

$$\begin{aligned} \iint_{R_2} f(x, y)\,\mathrm{d}A = &\int_{-2}^{-1}\int_{0}^{\sqrt{4-x^2}} f(x, y)\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} f(x, y)\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_{1}^{2}\int_{0}^{\sqrt{4-x^2}} f(x, y)\,\mathrm{d}y\,\mathrm{d}x \end{aligned}$$

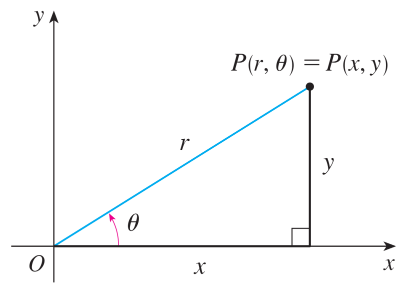
We Can Do Better

In short, integrating over each of these regions is a pain (and potentially even worse than that); we have nasty square roots to work with in both cases, and multiple regions to integrate over in the second case. But these are such simple regions! Something about this seems... wrong. It would be quite nice if there were a simpler way to proceed.

And there is! The key is that each region is nicely described using polar coordinates.

Polar Coordinates

Recall that every point P in $\mathbb{R}^2$ may be described in terms of its distance from the origin, r, and the signed angle that a line segment from the origin to P makes with the positive x-axis, θ:



r and θ make up the *polar coordinates* of the point P, whereas x and y are the *rectangular* or *Cartesian coordinates* of P. Recall also that we may convert between rectangular and polar coordinates using:

$$x^2 + y^2 = r^2, \quad x = r\cos(\theta), \quad y = r\sin(\theta)$$

Describing the Regions

Both of the regions mentioned at the outset of this section are straightforward to describe using polar coordinates. Indeed, the first is:

$$R_1 = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

and the second is:

$$R_2 = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$$

This looks much cleaner, indeed. But will it make integrating any friendlier? To find out, let's see how we can take advantage of this to compute integrals.

Integrating in Polar Coordinates

Theorem: Suppose that $f(x,y)$ is continuous on the region

$$R = \{(r,\theta) | a \le r \le b, \alpha \le \theta \le \beta\}$$

for where a, b, α, and β are all constants with $a \ge 0$ and $0 \le \beta - \alpha \le 2\pi$. Then

$$\iint_R f(x,y)\,\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))r\,\mathrm{d}r\,\mathrm{d}\theta$$

Note that since a, b, α, and β are all constants, Fubini's Theorem applies here, and the order of integration may be reversed.

Example

Evaluate $I = \iint_{R_2} (3x + 4y^2)\,\mathrm{d}A$ where R_2 is the region in the upper half plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, as given on the slide at the start of this section.

We know from above that we may describe R_2 as

$$R_2 = \left\{(r, \theta) | 1 \leq r \leq 2, 0 \leq \theta \leq \pi\right\}$$

Example, cont.

Therefore, we have:

$$\begin{aligned}
I &= \int_0^{\pi} \int_1^2 (3r\cos(\theta) + 4r^2\sin^2(\theta))r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\pi} \int_1^2 (3r^2\cos(\theta) + 4r^3\sin^2(\theta))\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\pi} \left(r^3\cos(\theta) + r^4\sin^2(\theta)\right)\Big|_{r=1}^{r=2} \mathrm{d}\theta \\
&= \int_0^{\pi} \left(7\cos(\theta) + 15\sin^2(\theta)\right) \mathrm{d}\theta \\
&= \int_0^{\pi} \left(7\cos(\theta) + \frac{15}{2}(1 - \cos(2\theta))\right) \mathrm{d}\theta \\
&= \left(7\sin(\theta) + \frac{15\theta}{2} - \frac{15}{4}\sin(2\theta)\right)\Big|_0^{\pi} = \boxed{\frac{15\pi}{2}}
\end{aligned}$$

An Important Note

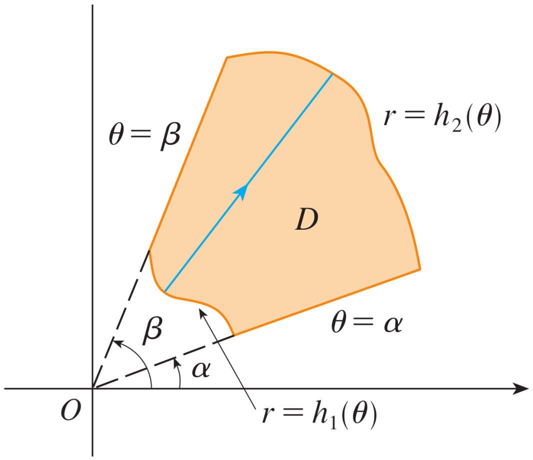
Above, we used one of the techniques you learned previously for integrating powers of sine and cosine functions. As you might imagine, these techniques will often be crucial to evaluating double integrals in polar coordinates, so be sure to review these methods as you work on the exercises for this section.

Is This New Technique Any Better?

I encourage you to try this example again using the description of R_2 in rectangular coordinates that we found earlier, or one that you worked out for yourself. See which you like better. Our new method isn't *always* more efficient, but for most cases where a region is most easily described in polar coordinates, it is.

Generalizing the Theorem

Of course, not all regions R can be described quite as nicely as above. For example, it may be that R is between two angles α and β, but that r is between two *functions* $h_1(\theta)$ and $h_2(\theta)$, as below:



A More General Result

In this case, we integrate as follows:

Suppose that $f(x, y)$ is continuous on a polar region D of the form:

$$D = \{(r, \theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

where α and β are constants. Then:

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta)) r\, dr\, d\theta$$

Since the inner bounds are not constants, Fubini's Theorem does **not** apply here! You *must* integrate in the order indicated.

Example

Find the (unsigned) volume V of the solid that lies between the surface $z = 1$ and the region D in the xy-plane bounded by the curve $x^2 + y^2 = 2x$.

First, notice that V can be computed with a double-integral:

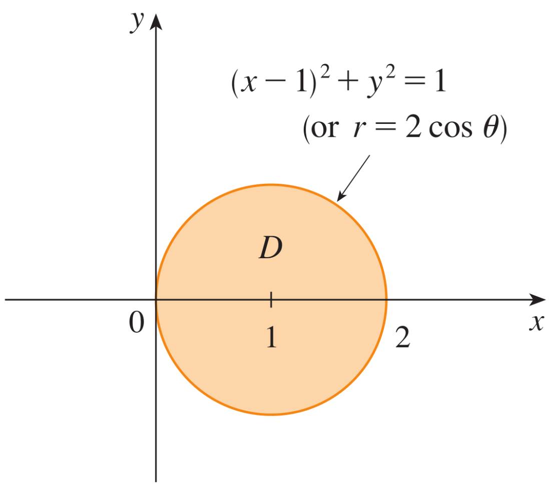
$$V = \iint_D 1\, dA$$

Now, let's sketch the region D. Completing the square, we have:

$$\begin{aligned} x^2 + y^2 = 2x &\Rightarrow x^2 - 2x + y^2 = 0 \\ &\Rightarrow (x-1)^2 - 1 + y^2 = 0 \\ &\Rightarrow (x-1)^2 + y^2 = 1 \end{aligned}$$

Example, cont.

Therefore, D is the region inside the circle of radius 1 centered at $(1, 0)$:



Example,cont.

Now that we know exactly what the region D looks like, we can turn to setting up $\iint_D 1\,\mathrm{d}A$ as an iterated integral. It is certainly possible to set up this iterated integral using rectangular coordinates (and I would encourage you to do so!), but given the shape of D, it would be wiser to try polar coordinates first.

We begin with the constant bounds on θ.

We want to know the smallest and largest values of θ which correspond to a point inside of D. I claim that every point inside D has a θ coordinate between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$. Indeed, every point in D (except the origin) is to the right of the y-axis; and on the other hand, there is at least one point inside of D for every angle between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, as we can see from the figure on the previous slide (We'll see another way to confirm this observation momentarily).

Example, cont.

Now for the bounds on r. We do so in much the same way as we did for type I and type II integrals in the previous section.

Fix an angle θ between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, and sketch a ray R that extends from the origin and makes an angle of θ with the positive x-axis. As we move out from the origin along R, we see that every point on R which is inside D has an r-coordinate between 0 and... whatever the r-coordinate of the point P_R, where R intersects the boundary circle of D, is.

How can we find the r-coordinate of this generic P_R?

This is where our conversions between rectangular and polar coordinates come in! Let's rewrite the equation $x^2 + y^2 = 2x$ in polar coordinates.

Example, cont.

We have:

$$\begin{aligned} x^2 + y^2 = 2x &\Rightarrow r^2 = 2r\cos(\theta) \\ &\Rightarrow r = 2\cos(\theta) \end{aligned}$$

Therefore, the r-coordinate of P_R is $2\cos(\theta)$.

Thus, given any θ between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, we see that the corresponding bounds on r are 0 and $2\cos(\theta)$. Therefore, we may describe D as:

$$D = \left\{ (r, \theta) \;\middle|\; \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 2\cos(\theta) \right\}$$

We are now in a position to set up our iterated integral.

Example, cont.

By the way, the equation $r = 2\cos(\theta)$ of our boundary circle can also help us confirm our bounds on θ. Indeed, note that this circle passes through the origin when $\theta = \frac{-\pi}{2}$, and again when $\theta = \frac{\pi}{2}$, and nowhere in-between, confirming our observation above.

Example, cont.

Putting everything together, we have:

$$\begin{aligned} V &= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} 1 \cdot r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 2\cos^2(\theta) \, \mathrm{d}\theta \\ &= 2\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos(2\theta)}{2} \right) \mathrm{d}\theta \\ &= \left(\theta + \frac{1}{2}\sin(2\theta) \right) \Bigg|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = \boxed{\pi} \end{aligned}$$

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Exercises

1. Evaluate $\iint_R 1 \, \mathrm{d}A$ where R is the region inside the unit circle in the xy-plane in two ways: first, using polar coordinates; and second, using rectangular coordinates [Note that you can also work out this double integral quickly by thinking of it as the volume of a familiar solid...].
2. Find the (unsigned) volume V_B inside the solid B in $\mathbb{R}^3$ bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$ [Hint: start by working out the intersection of the two surfaces].
3. Use a double integral to find the area A_ℓ enclosed by one loop ℓ of the four-leaved rose $r = \cos(2\theta)$ [Hint: Recall that the area of a region R is given by $\iint_R 1 \, \mathrm{d}A$].
4. Find the (unsigned) volume V of the solid in $\mathbb{R}^3$ that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Solutions

1. In polar coordinates, $\iint_R 1 \, \mathrm{d}A = \int_0^{2\pi} \int_0^1 r \, \mathrm{d}r \, \mathrm{d}\theta = \pi$.
   In rectangular coordinates, $\iint_R 1 \, \mathrm{d}A = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mathrm{d}y \, \mathrm{d}x = \pi$ where the outer integral may be computed using the substitution $x = \sin(\theta)$. Don't forget to change your bounds when using a substitution!
   Of course, $\iint_R 1 \, \mathrm{d}A$ is also the volume of the circular cylinder of radius 1 and height 1, which is π.
2. $V_B = \int_0^{2\pi} \int_0^1 (1 - r^2) r \, \mathrm{d}r \, \mathrm{d}\theta = \frac{\pi}{2}$
3. $A_\ell = \int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} r \, \mathrm{d}r \, \mathrm{d}\theta = \frac{\pi}{8}$
4. $V = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos(\theta)} r^3 \, \mathrm{d}r \, \mathrm{d}\theta = \frac{3\pi}{2}$