15.1: Double Integrals Over Rectangles

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Chapter Overview

In the previous chapter we learned the theory of differentiation for surfaces in $\mathbb{R}^3$. Since such surfaces arise as the graphs of two-variable functions, we in fact learned the theory of differentiation for such functions, with functions of more variables coming along as a free bonus. In this chapter we complete our study of the calculus of surfaces by exploring the theory of integration of two-variable functions. We'll also spend some time on the theory of integration of three-variable functions, but the theory of integration of functions of more variables won't come along quite as easily as it did for differentiation. That said, the foundational principles still apply for such functions, which will give us a basic idea of how to integrate them in special circumstances, as well.

We will first learn the theory of integration of two-variable functions and discuss mathematical applications of this theory, and then repeat the process for three-variable functions. Along the way we will learn two new coordinate systems for $\mathbb{R}^3$ which are useful on their own, but will also prove invaluable for evaluating such integrals.

Section Overview

In the first part of this section, we will recap the definition of the definite integral of a single-variable function, and then extend this idea to define the definite double integral of a two-variable function over a rectangle. We will then use this definition to estimate the value of such an integral.

Once we get a handle on this definition, we will then turn to the task of evaluating such definite integrals precisely (not just estimating them) by developing a new technique to do so: using so-called **iterated integrals**.

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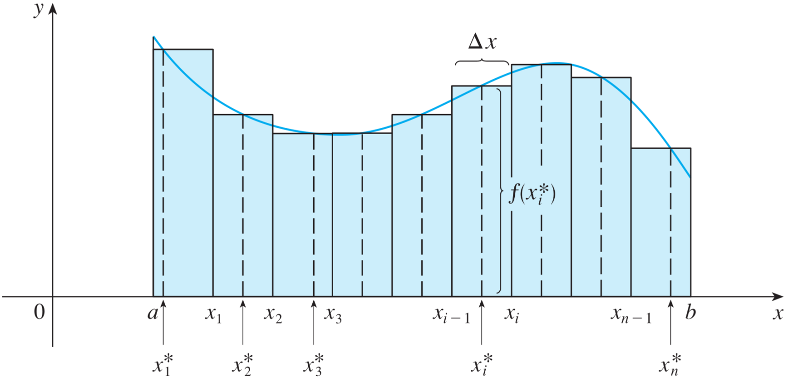
The Definite Integral

Recall that if we have a continuous function $f(x)$ on an interval $[a, b]$, we can estimate the signed area between the graph of $f(x, y)$ and the x-axis on this interval. We first divide $[a, b]$ into n subintervals of length Δx, and then choose test points x_i^* in each subinterval (typically the left- or right-hand endpoints of each subinterval). We then construct boxes above the ith subinterval of height $f(x_i^*)$. The area of each box is given by $f(x_i^*)\Delta x$, so that an estimate of the total area under the curve is given by adding up the areas of the boxes:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

(See the diagram on the next slide as further reminder)

The Definite Integral, cont.



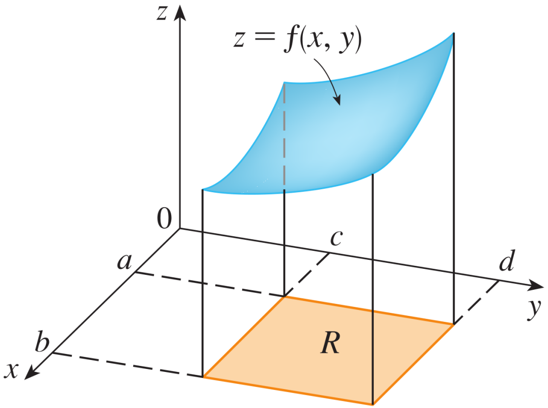
The Definite Integral, cont.

If we increase the number of boxes, our estimate improves. Thus, we *define* the definite integral as follows:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

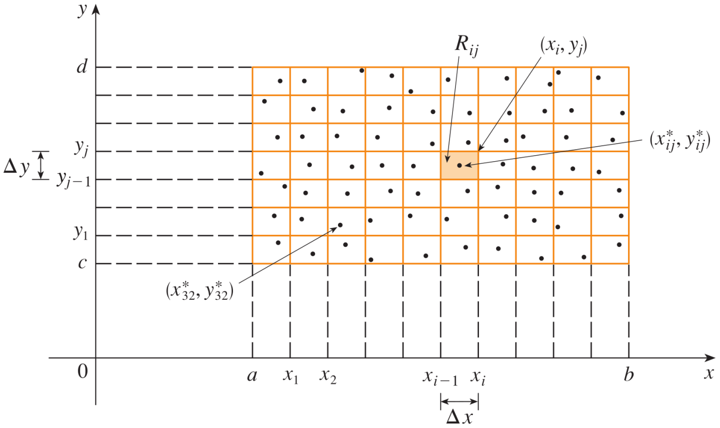
Multivariable Functions

Now let us turn to a function $f(x, y)$ of two variables. Suppose that $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$. Analogous to the single-variable case, we could then ask what the signed volume V between the graph of $f(x, y)$ and the xy-plane is:



Multivariable Functions, cont.

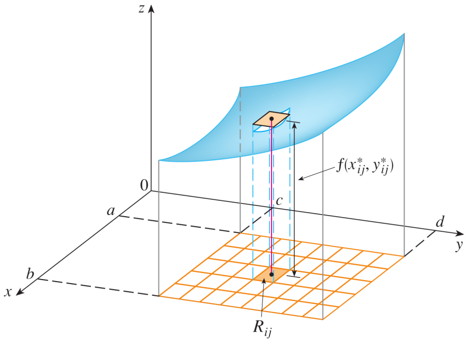
We proceed in much the same way as the single-variable case. We begin by dividing the rectangle R into subrectangles. We do this by breaking $[a, b]$ into m subintervals of length $\Delta x = (b - a)/m$, and $[c, d]$ into n subintervals of length $\Delta y = (d - c)/n$:



Multivariable Functions, cont.

We label each subrectangle R_{ij} for i and j between m and n respectively, as above. From each R_{ij} we choose a sample point (x_{ij}^*, y_{ij}^*), and then construct a box with base R_{ij} and uniform height $f(x_{ij}^*, y_{ij}^*)$. The volume of such a box is

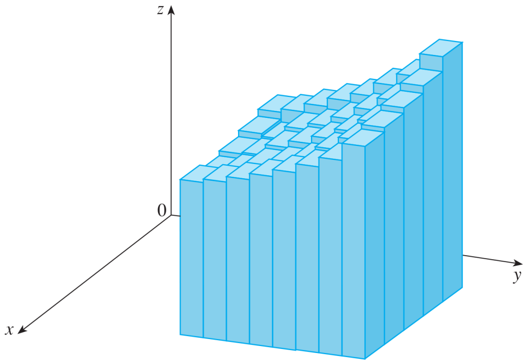
$$f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$$



Multivariable Functions, cont.

The total signed volume V between the rectangle R in the xy-plane and the graph of $f(x, y)$ can then be estimated by computing total signed volume inside all of these boxes:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta x \Delta y$$



Multivariable Functions, cont.

Now, as we let the number of boxes increase (i.e. as m and n increase), the estimate of V improves. Therefore, we see that:

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

(where ΔA is the product $\Delta x \Delta y$).

This volume is what we call the **double integral** of $f(x, y)$ over R, defined as follows, based on our work:

$$\iint_R f(x, y)\,\mathrm{d}A = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

Again, this gives the total signed volume in $\mathbb{R}^3$ between the rectangle R in the xy-plane and the graph of the function $f(x, y)$.

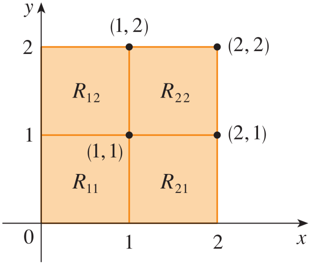
Example

Estimate the signed volume of the solid that lies between the square $R = [0, 2] \times [0, 2]$ and the elliptic paraboloid $z = 16 - x^2 - 2y^2$ by dividing R into four equal subsquares and choosing the sample point in each to be its upper-right corner.

First, note that by definition the signed volume of the described solid is

$$\iint_R (16 - x^2 - 2y^2)\,\mathrm{d}A$$

Now, let's sketch R, its subsquares, and the sample points:



Example, cont.

For convenience, let's give the integrand a name:

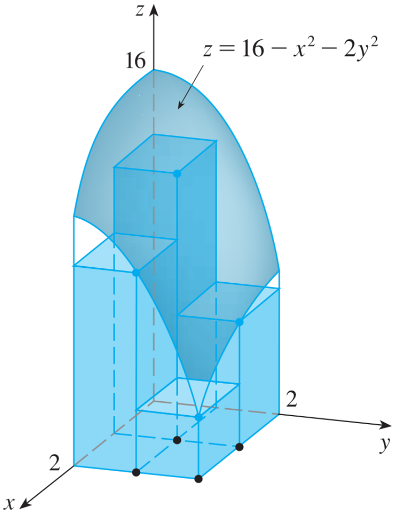
$$f(x, y) := 16 - x^2 - 2y^2$$

Each subsquare R_{ij} has length and width 1, for a total base area of 1. Therefore, the signed volume between the graph of $f(x, y)$ and a given subrectangle R_{ij} is approximately $1 \cdot f(x_{ij}^*, y_{ij}^*)$, where (x_{ij}^*, y_{ij}^*) is the point in the upper-right corner of R_{ij}. Thus, $\iint_R (16 - x^2 - 2y^2)\, \mathrm{d}A$ can be approximated by adding up these estimates, as follows:

$$\begin{aligned}\iint_R (16 - x^2 - 2y^2)\, \mathrm{d}A &\approx f(1,1) \cdot 1 + f(1,2) \cdot 1 \\ &\qquad + f(2,1) \cdot 1 + f(2,2) \cdot 1 \\ &= 13 + 7 + 10 + 4 = \boxed{34}\end{aligned}$$

Example, cont.

Here is a picture of what we have just described:



Can We Do Better?

We now know how to estimate the value of a double integral of a two-variable function over a rectangle, and, believe it or not, this is quite an important skill. In practice, many integrals (single, double, or otherwise) cannot be computed precisely in a nice, closed form, so approximation is a commonly-used tool (hence our devoting some time to learning this skill). In fact, it's important enough that there are entire courses which expand on this idea and related problems: courses in what is called numerical analysis.

However, in many problems we'd often like to do better than an estimate, if we can. Thus, for the remainder of this and subsequent sections in this chapter, we will learn to evaluate double integrals precisely using what are called **iterated integrals**.

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The Setup

If $f(x, y)$ is a continuous function on the rectangular region $R = [a, b] \times [c, d]$ in xy-plane, and we wish to calculate the net signed volume between R and the graph of $f(x, y)$, we learned above that we call this volume the **double integral of** $f(x, y)$ **over** R, and can evaluate it as follows:

$$\iint_R f(x, y)\, \mathrm{d}A = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

Therefore, calculating net signed volume between R and the graph of $f(x, y)$ amounts to evaluating the limit on the right. If you think this looks like a total nightmare, in general it absolutely is. So, we would like to find something more efficient. That's our next goal.

Partial Integrals

Let's begin with some notation. We write:

$$\int_c^d f(x, y)\,\mathrm{d}y$$

to mean that we hold x fixed and evaluate the definite integral of $f(x, y)$ with respect to y in the usual manner from single-variable calculus, for y between c and d. We call this the **partial integral of** $f(x, y)$ **with respect to** y. We denote the partial integral of $f(x, y)$ with respect to x similarly, with $\mathrm{d}x$ instead of $\mathrm{d}y$.

Notice that there is a strong analogy between the partial derivative of $f(x, y)$ with respect to x or y, and these partial integrals.

Example

Evaluate $\int_1^3 (x^2 + 2xy + y^2)\,\mathrm{d}y$.

Remember: we think of x as a constant here and evaluate this definite integral with respect to y. Thus, by definition we have:

$$\int_1^3 (x^2 + 2xy + y^2)\,\mathrm{d}y = \left(x^2 y + xy^2 + \frac{y^3}{3}\right)\Bigg|_{y=1}^{y=3}$$

$$= (3x^2 + 9x + 9) - \left(x^2 + x + \frac{1}{3}\right)$$

$$= \boxed{2x^2 + 8x + \frac{26}{3}}$$

Iterated Integrals

With this in mind, let us to return to our original problem. To evaluate $\iint_R f(x,y)\,\mathrm{d}A$ where $R = [a,b] \times [c,d]$, it turns out that we can compute it as an **iterated integral**, i.e., as a pair of partial integrals, working from the inside out:

$$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_a^b \left[\int_c^d f(x,y)\,\mathrm{d}y\right]\mathrm{d}x$$

Example

Evaluate $\iint_R x^2 y \, \mathrm{d}A$ for $R = [1,2] \times [0,3]$.

From the previous slide, we have:

$$\begin{aligned}
\iint_R x^2 y \, \mathrm{d}A &= \int_1^2 \left[\int_0^3 x^2 y \, \mathrm{d}y \right] \mathrm{d}x \\
&= \int_1^2 \left[x^2 \frac{y^2}{2} \Bigg|_{y=0}^{y=3} \right] \mathrm{d}x \\
&= \int_1^2 \frac{9x^2}{2} \, \mathrm{d}x \\
&= \frac{3x^3}{2} \Bigg|_1^2 = \boxed{\frac{21}{2}}
\end{aligned}$$

Key Point

Let's evaluate the iterated integral from the previous slide in the opposide order:

$$\int_0^3 \left[\int_1^2 x^2 y \, dx\right] dy = \int_0^3 \left[\frac{x^3}{3} y \Bigg|_{x=1}^{x=2}\right] dy$$
$$= \int_0^3 \frac{7y}{3} \, dy$$
$$= \frac{7y^2}{6}\Bigg|_0^3 = \boxed{\frac{21}{2}}$$

We get the same result! This is no accident.

Fubini's Theorem

Fubini's Theorem: Suppose that $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$. Then:

$$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Fubini's Theorem means that you can choose the order of integration, which can be quite handy in some problems!

Example

Evaluate $\iint_R y \sin(xy)\, \mathrm{d}A$ where $R = [1,2] \times [0,\pi]$.

Let's try integrating with respect to y first, as we first learned to do. We have:

$$\iint_R y \sin(xy)\, \mathrm{d}A = \int_1^2 \int_0^\pi y \sin(xy)\, \mathrm{d}y\, \mathrm{d}x$$

To complete the integral in this order, we would have to use integration by parts. In fact, we would have to use it twice (try it!).

Example, cont.

However, Fubini's theorem tells us that we are free to choose the order of integration! If we integrate with respect to x first, the problem is much more straightforward:

$$
\begin{aligned}
\iint_R y\sin(xy)\,\mathrm{d}A &= \int_0^{\pi}\int_1^2 y\sin(xy)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{\pi} -\cos(xy)\Big|_{x=1}^{x=2}\,\mathrm{d}y \\
&= \int_0^{\pi} \big(-\cos(2y)+\cos(y)\big)\,\mathrm{d}y \\
&= \left[\frac{-1}{2}\sin(2y)+\sin(y)\right]\Bigg|_0^{\pi} = \boxed{0}
\end{aligned}
$$

Average Value

The last thing we wish to define is the average value of a function $f(x, y)$ on a rectangle R in the xy-plane. To get us to the proper definition, let's first think carefully about what averages are.

Suppose we have a list of numbers: $9, 8, 9$, and 6. The average value of this list is

$$\frac{9+8+9+6}{4} = 8$$

What this means is that if every number in the list were 8, the sum of the numbers on the list would not change. So, for example, if these were scores on quizzes graded out of 10, the student's average score is $8/10$, i.e., if they had gotten $8/10$ on all four quizzes, they would, in some sense, be doing just as well as with their actual numerical grades. Averaging is, therefore, one way of smoothing out the noise in a list of data (though not the best way in all circumstances!).

Average Value, cont.

How does this relate to the average value of a function? Well, we could think of our list on the previous page as a discrete function $f(x)$, with domain $\{1, 2, 3, 4\}$, and $f(1) = 9$, $f(2) = 8$, $f(3) = 9$, and $f(4) = 6$. Then the average value, f_{avg}, of $f(x)$ on its domain is, again, 8:

$$f_{\text{avg}} = \frac{f(1) + f(2) + f(3) + f(4)}{4} = 8$$

That is, if $f(x) = 8$ for every x in the domain of $f(x)$, then the sum of all the values of $f(x)$ wouldn't change.

But what if $f(x)$ is continuous? For example, if $f(x)$ is continuous on the interval $[a, b]$, how could we compute its average value on this domain? We certainly can't just "add up" every value of $f(x)$ on this interval and then divide by the number of values... there's infinitely-many of each! But there is a sense in which we can.

Average Value, cont.

This is where the definite integral comes in. In some sense,

$$\int_a^b f(x)\,\mathrm{d}x$$

is what you get when you “add up” all the values of $f(x, y)$ on the interval $[a, b]$, and the length of this interval, $b - a$, is the “number of values.” Therefore, we define:

$$f_{\text{avg}} := \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}$$

The effect is the same as in the discrete case: if $f(x)$ were equal to the constant f_{avg} on the entire interval $[a, b]$, then the “sum” $\int_a^b f(x)\,\mathrm{d}x$ wouldn't change.

Average Value, cont.

With this in mind, how should we define the average value of a two-variable function $f(x, y)$ on a rectangle R? Well, the same way as above! In a sense,

$$\iint_R f(x, y)\,\mathrm{d}A$$

is what we get when we "add up" all the values of $f(x, y)$ on R, and the total number of values is the area of the rectangle. Therefore, we define the **average value** of a function $f(x, y)$ on a rectangle R as:

$$f_{\text{avg}} = \frac{1}{A(R)} \iint_R f(x, y)\,\mathrm{d}A$$

where $A(R)$ is the area of the rectangle R.

Example

So, for example, we previously found that

$$\iint_R x^2 y \, \mathrm{d}A = \frac{21}{2}$$

where R is the rectangle $[1,2] \times [0,3]$. Therefore, the average value of $f(x,y) := x^2 y$ on this rectangle is:

$$f_{\text{avg}} = \frac{1}{A(R)} \iint_R f(x,y) \, \mathrm{d}A = \frac{1}{1 \cdot 3} \cdot \frac{21}{2} = \boxed{\frac{7}{2}}$$

I'll leave it to you to check that the double integral of $x^2 y$ over R is the same as the double integral of f_{avg} over R.

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Exercises

1. Estimate $\iint_R (x - 3y^2)\,\mathrm{d}A$ where

$$R = [0,2] \times [1,2] = \left\{(x,y) \mid 0 \le x \le 2, 1 \le y \le 2\right\}$$

   as follows: divide R into four equal rectangles by dividing the intervals $[0,2]$ and $[1,2]$ in half, and constructing rectangles from these divisions in the obvious way. Then, to carry out your estimate, let the sample point in each rectangle be its upper-right corner.
2. Repeat the previous exercise, this time letting the sample points be the lower left corner of each rectangle.
3. Repeat the first exercise, this time letting the sample points be the center of each rectangle (this is called the *Midpoint Rule*, akin to the rule of the same name for single-variable functions).

Exercises, cont.

4. Evaluate $\iint_R x^2 y \, \mathrm{d}A$ where $R = [0,3] \times [1,2]$. Then calculate the average value of $h(x,y) := x^2 y$ on R.
5. Calculate the signed volume of the solid that lies between the graph of $f(x,y) = x - 3y^2$ and the rectangle $D = [0,2] \times [1,2]$ in the xy-plane.
6. Find the volume of the solid S bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the coordinate planes [Hint: Begin by writing z as a function of x and y, then express the volume of the solid as a double integral over a rectangle].

Solutions

1. $\iint_R (x - 3y^2)\,\mathrm{d}A \approx \frac{-63}{4} = -15.75$
2. $\iint_R (x - 3y^2)\,\mathrm{d}A \approx \frac{-35}{4} = -8.75$
3. $\iint_R (x - 3y^2)\,\mathrm{d}A \approx \frac{-95}{8} = -11.875$
4. $\iint_R x^2 y\,\mathrm{d}A = \frac{27}{2}$, and $h_{\text{avg}} = \frac{9}{2}$.
5. The signed volume of the solid that lies between the graph of $f(x, y)$ and the rectangle D is $\iint_D (x - 3y^2)\,\mathrm{d}A = -12$
6. The volume of S is $\int_0^2 \int_0^2 (16 - x^2 - 2y^2)\,\mathrm{d}y\,\mathrm{d}x = 48$ [there are certainly other iterated integrals that would give us the same volume].