14.7: Maximum and Minimum Values

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Overview

We now know a lot about multivariable functions: how to evaluate them; how to calculate their limits; how to take their derivatives in any direction; and how to estimate their values using tangent planes. Mimicking our work in single-variable calculus, in the last two sections of this chapter we turn to optimization, i.e., maximizing and minimizing such functions.

We will begin by working out how to find the local and absolute extrema (maxima and minima) of two-variable functions by generalizing the concept of critical points to three dimensions. We will then turn to optimization problems.

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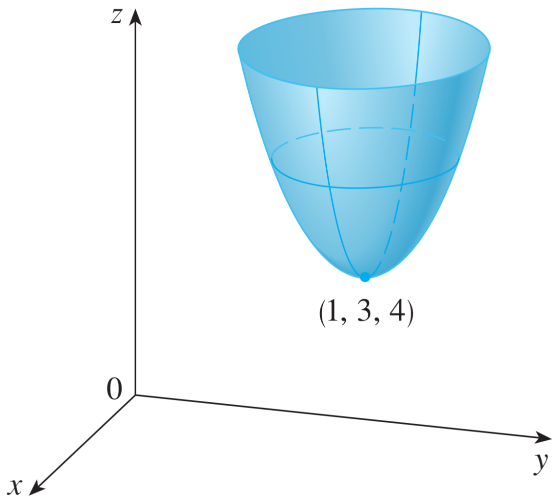
Local Maxima and Minima

A function $f(x, y)$ has a **local maximum** (resp. **local minimum**) at (a, b) if $f(x, y) \leq f(a, b)$ (resp. $\geq f(a, b)$) for all (x, y) close to (a, b). $f(a, b)$ is called a **local maximum** (resp. **minimum**) **value**.

The Picture

Here, $f(1,3)$ is a local minimum of f, with local minimum value $4 = f(1,3)$.



Critical Points

Recall that if a single-variable function $f(x)$ has a local maximum or minimum at $x = a$, then $f'(a) = 0$ or $f'(a)$ does not exist. If either of the latter is true, we call $x = a$ a *critical number* of $f(x)$. A similar result is true for two-variable functions. We begin with a definition.

Definition: The point (a,b) is called a **critical point** of the function $f(x,y)$ if both $f_x(a,b) = 0$ *and* $f_y(a,b) = 0$, or if either of $f_x(a,b)$ or $f_y(a,b)$ does not exist.

Example

Find the critical points of $f(x,y) = x^2 + y^2 - 2x - 6y + 14$.

By the previous slide, a point P is a critical point of $f(x,y)$ if either one of the partial derivatives of $f(x,y)$ does not exist at P, or if *both* of these partial derivatives are zero at P. Therefore, we begin by computing the partial derivatives of $f(x,y)$:

$$f_x(x,y) = 2x - 2 \qquad \text{and} \qquad f_y(x,y) = 2y - 6$$

Since $f_x(x,y)$ and $f_y(x,y)$ are polynomials, they exist at every point in $\mathbb{R}^2$. On the other hand, by inspection we see that $f_x(x,y) = 0$ when $x = 1$ and $f_y(x,y) = 0$ when $y = 3$. Therefore, given the requirement that both partial derivatives must be zero at the same time at a critical point, we see that the lone critical point of $f(x,y)$ is $\boxed{(1,3)}$.

A Key Result

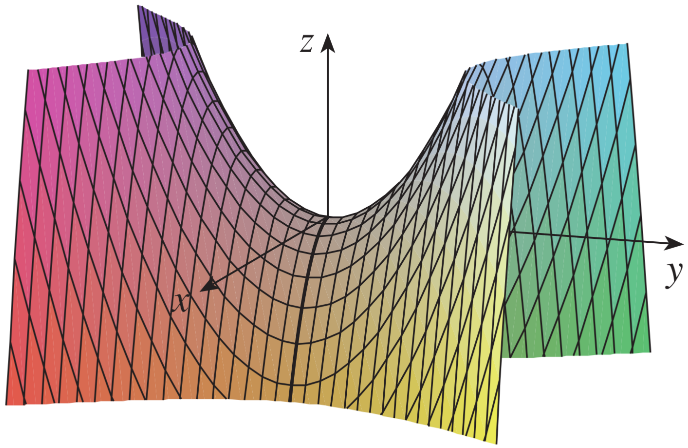
The utility of critical points is completely analogous to the role of critical numbers for single-variable functions.

Theorem: If $f(x,y)$ has a local maximum or minimum at (a,b), then f has a critical point there.

To extend the analogy even further, when a single-variable function $f(x)$ has critical number $x = a$, the slope of the line tangent to f is horizontal (or the tangent line does not exist). When a two-variable function $f(x,y)$ has a critical point at (a,b), the plane tangent to the graph of f is flat.

Important Note

Just as in the single-variable case, having a critical point at (a, b) does not automatically mean that the function has a local maximum or minimum there. For example, consider the critical point at the origin of this function:



Saddle Points

A **saddle point** is a critical point of $f(x, y)$ that corresponds to neither a local minimum or local maximum of f.

A Key Question

To summarize, local minima and maxima must occur at critical points, but not every critical point is a local minimum or local maximum.

The question that now confronts us is: given a set of critical points, how can we decide if each is a local minimum, local maximum, or neither? The answer lies in the Second Derivative Test.

The Second Derivative Test

Suppose that (a, b) is a critical point of $f(x, y)$ and that the second partial derivatives of f are continuous near (a, b). Let:

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

If:

1. $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
2. $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
3. $D(a, b) < 0$, then (a, b) is a saddle point of f.
4. $D(a, b) = 0$, no conclusion may be drawn from this test.

To recall the formula for $D(a, b)$, it helps to think of it as a determinant:

$$D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}$$

Example

Does the critical point $(1,3)$ of $f(x,y) = x^2 + y^2 - 2x - 6y + 14$ that we discovered above correspond to a local minimum value of $f(x,y)$, a local maximum value of $f(x,y)$, or neither?

The second derivative test from the previous slide will likely answer this question for us. Therefore, we begin by finding the necessary ingredients for this test:

$$\begin{aligned} f_x(x,y) = 2x - 2 \quad &\Longrightarrow \quad f_{xx}(x,y) = 2 \text{ and } f_{xy}(x,y) = 0 \\ &\Longrightarrow \quad f_{xx}(1,3) = 2 \text{ and } f_{xy}(1,3) = 0 \\ \text{and} \quad & \\ f_y(x,y) = 2y - 6 \quad &\Longrightarrow \quad f_{yy}(x,y) = 2 \\ &\Longrightarrow \quad f_{yy}(1,3) = 2 \end{aligned}$$

Example, cont.

Now we apply the second derivative test to the point $(1,3)$:

$$\begin{aligned} D(1,3) &= f_{xx}(1,3)f_{yy}(1,3) - [f_{xy}(1,3)]^2 \\ &= 2 \cdot 2 - 0^2 \\ &= 4 - 0 = 4 \end{aligned}$$

Since $D(1,3) > 0$ and $f_{xx}(1,3) = 2 > 0$, by the second derivative test we have that $\boxed{f(1,3) = 4 \text{ is a local minimum value of } f(x,y).}$

Example

Find the local minimum values, local maximum values, and the saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

We know from above that local minima, local maxima, and saddle points of $f(x,y)$ *must* occur at critical points of $f(x,y)$. Therefore, we begin by finding these critical points, as we did previously:

$$f_x(x,y) = 4x^3 - 4y \qquad \text{and} \qquad f_y(x,y) = 4y^3 - 4x$$

Since both of these partial derivatives are polynomials, they are defined on all of $\mathbb{R}^2$. Therefore, the critical points of $f(x,y)$ must occur at the points where both partial derivatives are zero *at the same time*. In other words, we want to solve the system of equations:

$$4x^3 - 4y = 0 \tag{1}$$

$$4y^3 - 4x = 0 \tag{2}$$

Take a moment, and see if you can solve this system.

Example, cont.

Here's my solution to the system of equations on the previous slide. First, note that:

$$(1) \Rightarrow y = x^3$$

Therefore, plugging this information into the second equation, we obtain:

$$\begin{aligned} (2) \text{ and } y = x^3 &\Rightarrow 4(x^3)^3 - 4x = 0 \\ &\Rightarrow x^9 - x = 0 \\ &\Rightarrow x(x^8 - 1) = 0 \\ &\Rightarrow x = 0, \ x = 1, \ \text{or } x = -1 \end{aligned}$$

Therefore, if (1) and (2) are true at the same time, we must have that $x = 0$ or $x = \pm 1$. But these aren't critical points, they're just x-values. How can we get from these to critical points? Take a moment to see if you can work this out.

Example, cont.

Let's find the critical points. We know from above that at any critical point, we must have $y = x^3$. Therefore:

$$y = x^3 \text{ and } x = 0 \Rightarrow y = 0$$
$$y = x^3 \text{ and } x = 1 \Rightarrow y = 1$$
$$y = x^3 \text{ and } x = -1 \Rightarrow y = -1$$

Therefore, the critical points of $f(x, y)$ are $(0, 0)$, $(1, 1)$, and $(-1, -1)$ (By the way, you can quickly check that each of these truly is a critical point of $f(x, y)$; how?).

Example, cont.

Now that we have the critical points, we use the Second Derivative Test to classify them. Let's begin by finding the ingredients for the test:

$$f_x(x, y) = 4x^3 - 4y \quad \Longrightarrow \quad f_{xx}(x, y) = 12x^2 \text{ and } f_{xy}(x, y) = -4$$
$$f_y(x, y) = 4y^3 - 4x \quad \Longrightarrow \quad f_{yy}(x, y) = 12y^2$$

Now we approach each critical point in turn. Let's begin with $(0, 0)$:

$$\begin{aligned} D(0,0) &= f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 \\ &= 0 \cdot 0 - (-4)^2 = 0 - 16 = -16 < 0 \end{aligned}$$

Therefore, by the second derivative test, $(0, 0)$ is a saddle point of $f(x, y)$. A similar calculation shows that $f(-1, -1) = -1$ and $f(1, 1) = -1$ are local minimum values of $f(x, y)$.

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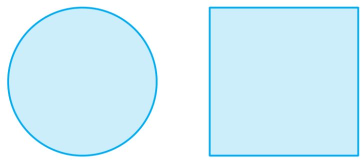
The Setup

Up to this point, we have talked exclusively about local maxima and minima, but this raises an interesting related question: does a function have a *global* or *absolute* maximum or minimum?

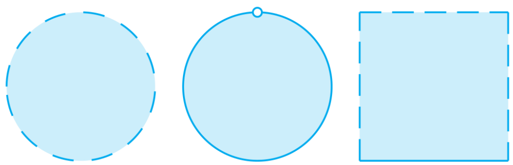
In general, of course, the answer is no; most functions are not bounded above or below. However, there are situations in which functions are guaranteed to have absolute maxima and minima, and that is what we'll talk about now.

Closed Sets

A **closed set** in $\mathbb{R}^2$ is a set of points that includes its boundary. Here are some examples:



(a) Closed sets



(b) Sets that are not closed

The Key Result

Theorem (Extreme Value Theorem for Functions of Two Variables): If $f(x, y)$ is continuous on a closed, bounded set A in $\mathbb{R}^2$, then f attains an absolute minimum and absolute maximum value in A.

Finding Absolute Minima and Maxima

Now we know for sure that a continuous $f(x, y)$ has an absolute minimum and maximum on a closed, bounded set D. But how do we find them?

Well, they must occur either in the interior of A or on its boundary. Local extrema on the interior of A *must* occur at critical points, for the same reasons that local extrema of $f(x, y)$ occur at critical points when we consider $f(x, y)$ globally. To find extrema on the boundary of A, we will utilize the closed-interval test for extrema from single-variable calculus.

Let us first formally state this method, and then see how it works with an example.

Finding Absolute Minima and Maxima, cont.

To find the absolute minimum and maximum values of a continuous function $f(x, y)$ on a closed, bounded set A:

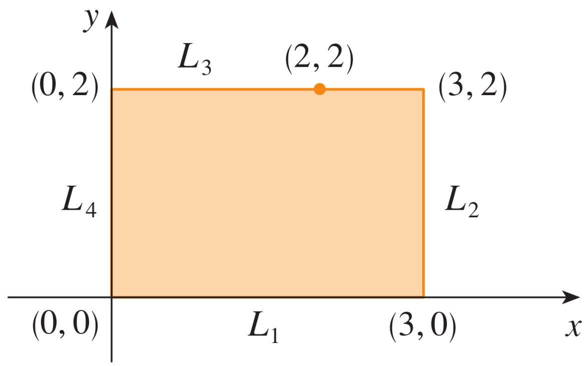
1. Find the values of $f(x, y)$ at the critical points of $f(x, y)$ in A;
2. On each boundary component of A, $f(x, y)$ can always be thought of as a single-variable function, due to the consistent relationship between x and y on such a component. Furthermore, there will always be strict restrictions on the values of x, y, or both on each boundary segment, given that A is a bounded set. Therefore, to find the extreme values of $f(x, y)$ on the boundary of A, we use the closed interval test from single-variable calculus on each of these boundary segments;
3. The absolute maximum and minimum values of $f(x, y)$ on A are the largest and smallest of the values of $f(x, y)$ obtained in the previous two steps, respectively.

Example

Find the absolute minimum and maximum values of $f(x, y) = x^2 - 2xy + 2y$ on the region R bounded by the rectangle

$$\{(x, y) \mid 0 \le x \le 3, 0 \le y \le 2\} = [0, 3] \times [0, 2]$$

We will proceed as indicated on the previous slide. To help us get started, let us begin by drawing R:



Example, cont.

Now, let's find the critical points of $f(x, y)$ in exactly the same way we did previously. We have:

$$f_x(x, y) = 2x - 2y \qquad \text{and} \qquad f_y(x, y) = -2x + 2$$

Notice that $f_x(x, y)$ and $f_y(x, y)$ are both polynomials, so they are defined at every point in $\mathbb{R}^2$. Therefore, critical points of $f(x, y)$ can occur only when *both* $f_x(x, y) = 0$ and $f_y(x, y) = 0$.

By inspection, $f_y(x, y) = 0$ when $x = 1$ and $f_y(x, y) = 0$ when $x = y$. Therefore, both partial derivatives are zero at the same time only at the point $(1, 1)$, the lone critical point of $f(x, y)$. Notice that $(1, 1)$ is, indeed, inside R. Therefore, as suggested in the method we are following, we evaluate:

$$f(1, 1) = 1$$

Example, cont.

Next we must find the extrema of $f(x, y)$ on the boundary of R. As indicated, we will always be able to use the closed-interval test from single-variable calculus to help us. Let's see how this works.

Let's begin by seeking the maximum and minimum values of $f(x, y)$ on the boundary segment L_1 of R. L_1 is the line segment $y = 0$, where $0 \le x \le 3$. Therefore, when we restrict the domain of $f(x, y)$ to just this one line segment, we see that, on L_1, we have:

$$f(x, y) = f(x, 0) = x^2$$

with $0 \le x \le 3$. Therefore, what we are looking to find are the absolute maximum and minimum values of the single-variable function $g_1(x) := x^2$ on the interval $[0, 3]$. Now we see the closed-interval test emerging!

Example, cont.

As a reminder, the closed-interval test from single-variable calculus tells us that if we wish to find the absolute maximum and minimum values of the function $g(x)$ on the interval $[a, b]$, then we evaluate $g(x)$ at $x = a$, $x = b$, and any critical numbers of $g(x)$ on the interval $[a, b]$. The largest value returned is the absolute maximum value of $g(x)$ on $[a, b]$, and the smallest value returned is the absolute minimum of $g(x)$ on $[a, b]$.

Let's apply this test to the current situation.

Example, cont.

Remember, we're trying to find the absolute minimum and maximum values of $g_1(x) := x^2$ on the interval $[0, 3]$. We begin by finding any critical numbers of $g_1(x)$ on $[0, 3]$. We have:

$$g_1'(x) = 2x$$

Since $g_1'(x)$ is a polynomial, it is defined everywhere, so that the only critical number of $g_1(x)$ occurs when $g_1'(x) = 0$, i.e. when $x = 0$, by inspection.

Evaluating $g_1(x)$ at the critical number $x = 0$ and at the endpoints of the interval $[0, 3]$, we obtain:

$$g_1(0) = 0 \qquad \text{and} \qquad g_1(3) = 9$$

Therefore, along L_1, $f(x, y)$ (that is to say, $g_1(x)$) has an absolute minimum value of $f(0, 0) = g_1(0) = 0$, and an absolute maximum value of $f(3, 0) = g_1(3) = 9$.

Example, cont.

We proceed in a nearly identical way along L_2, L_3, and L_4, and find that:

- Along L_2, $f(x, y)$ has an absolute minimum value of $f(3, 2) = 1$ and an an absolute maximum value of $f(3, 0) = 9$;
- Along L_3, $f(x, y)$ has an absolute minimum value of $f(2, 2) = 0$ and an an absolute maximum value of $f(0, 2) = 4$;
- Along L_4 $f(x, y)$ has an absolute minimum value of $f(0, 0) = 0$ and an an absolute maximum value of $f(0, 2) = 4$.

(Take some time to verify these!)

Therefore, comparing the values of $f(x, y)$ that we found at the critical points of $f(x, y)$ in R and on the boundary of R, We find at last that $f(x, y)$ has an absolute maximum value of $f(3, 0) = 9$ and an absolute minimum value of $f(0, 0) = f(2, 2) = 0$ on R.

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Optimization

In our final extension of maximum/minimum problems from single-variable calculus, we turn at last to optimization.

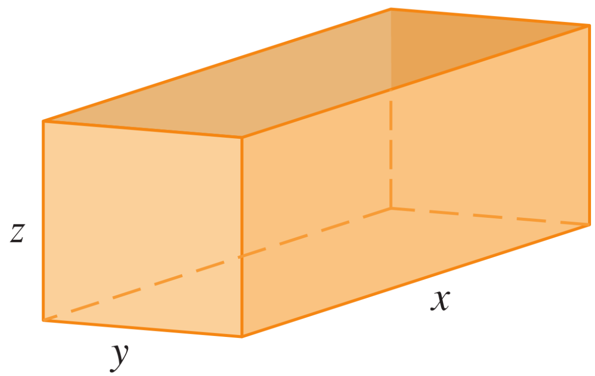
You may recall from single-variable calculus that optimization problems essentially boil down to finding the absolute maximum or minimum value of a two-variable function, given some constraint on those variables. For example, a couple of common problems include finding the largest area of a rectangle with a given perimeter; or the smallest sum of a pair of numbers whose product is a given number.

Here in multivariable calculus, we will be doing essentially the same thing — calculating the absolute maximum or minimum value of a three-variable function, subject to a constraint on those variables — in almost exactly the same way! Let's dive right in.

Example

A rectangular box without a lid is to be made from 12m^2 of cardboard. Find the maximum volume of such a box.

We will proceed largely by imitating the procedure from single-variable calculus. With that in mind, let us begin by drawing the box:



Example, cont.

We wish to calculate the maximum volume of this box. Given our labels in the diagram, this means that we wish to find the maximum value of the three-variable function:

$$V(x, y, z) := xyz$$

Of course, as things stand now, this function has no maximum! Indeed, if we were to let x, y, and z grow without bound, $V(x, y, z)$ would also grow arbitrarily large.

This is where our constraint comes in. The restriction on the variables given in the statement of the problem is what will allow a maximum value to appear. In this case, we are told that, first, the box on the previous slide has no top; and second, this box is made out of 12m^2 of material. In other words, the surface area of the box must be 12m^2. These together give us our constraint on x, y, and z:

$$xy + 2yz + 2xz = 12$$

Example, cont.

We now need to incorporate this constraint into our function somehow. The most straightforward way to do so is as follows: We begin by solving the constraint equation for one of the variables (in this case, I have chosen z, although any of the three variables would be fine in this case):

$$xy + 2yz + 2xz = 12 \;\Rightarrow\; z = \frac{12 - xy}{2y + 2x}$$

and then we substitute $z = \frac{12-xy}{2y+2x}$ in for z in the function $V(x, y, z)$, to obtain:

$$V(x, y, z) = xyz \text{ and } z = \frac{12 - xy}{2y + 2x}$$
$$\Rightarrow \; V(x, y) = \frac{xy(12 - xy)}{2y + 2x} = \frac{12xy - x^2y^2}{2y + 2x}$$

All of this done, we are now in a position to compute the absolute maximum value of $V(x, y)$ using the methods we have learned in this section.

Example, cont.

Now's a good time to take a quick look at what values x and y can take on. Notice that x and y are both lengths, and therefore both must be positive numbers. Furthermore, while there are genuine engineering constraints on what the length and the width of the cardboard box can realistically be, there's no *numerical* constraint, so that x and y can both be any positive numbers at all, as far as we're concerned.

Therefore, we are looking for the absolute maximum value of of $V(x, y)$ on the set

$$B := \left\{(x, y) \in \mathbb{R}^2 \mid x, y > 0\right\}$$

Since B is neither closed nor bounded, our absolute maximum value must appear at a critical point of $V(x, y)$.

Example, cont.

Hiding some work, we have:

$$V_x(x,y) = \frac{y^2(12-2xy-x^2)}{2(x+y)^2}, \qquad V_y(x,y) = \frac{x^2(12-2xy-y^2)}{2(x+y)^2}$$

Recall that a critical point of $V(x,y)$ occurs when either $V_x(x,y)$ or $V_y(x,y)$ is undefined; or when $V_x(x,y)$ and $V_y(x,y)$ are both zero at the same time.

This is the first time that we have encountered a case where $V_x(x,y)$ and $V_y(x,y)$ aren't defined on all of $\mathbb{R}^2$! By looking at the denominators of both partial derivatives, we see that both are undefined whenever $x+y=0$ — that is, both partial derivatives are undefined at infinitely-many points! However, notice that:

$$x+y=0 \;\Rightarrow\; x=-y$$

Therefore, since x and y must both be positive numbers, none of these critical points of $V(x,y)$ occur in B, so we can remove them all from consideration.

Example, cont.

Therefore, the critical point(s) we are interested in must occur when $V_x(x,y)$ and $V_y(x,y)$ are zero at the same time. Let's begin with $V_x(x,y)$:

$$\begin{aligned}
&\frac{y^2(12-2xy-x^2)}{2(x+y)^2} = 0 \\
\Rightarrow\;& y^2(12-2xy-x^2) = 0 \\
\Rightarrow\;& y^2 = 0 \text{ or } 12-2xy-x^2 = 0 \\
\Rightarrow\;& y = 0 \text{ or } 12-2xy-x^2 = 0
\end{aligned}$$

In a nearly identical calculation, setting $V_y(x,y)$ equal to zero gives that $x=0$ or $12-2xy-y^2=0$.

Example, cont.

Now, because we are only interested in critical points where x and y are both positive, we can throw out the cases $x=0$ and $y=0$ from above. Therefore, a critical point for $V(x,y)$ occurs in B exactly when the following system of equations is true:

$$12-2xy-x^2=0 \tag{1}$$
$$12-2xy-y^2=0 \tag{2}$$

There are many ways to solve this system, but here's one that I find particularly clean. Note that:

$$(1) \;\Rightarrow\; x^2 = 12-2xy \qquad \text{and} \qquad (2) \;\Rightarrow\; y^2 = 12-2xy$$

Therefore,

$$\begin{aligned}
(1) \text{ and } (2) \;\Rightarrow\;& x^2 = y^2 \\
\Rightarrow\;& x = \pm y \\
\Rightarrow\;& x = y
\end{aligned}$$

where the final equation holds since x and y must both be positive numbers.

Example, cont.

To finish solving the system, notice that:

$$\begin{aligned}
(1) \text{ and } x=y \;\Rightarrow\;& 12-2x^2-x^2=0 \\
\Rightarrow\;& 12-3x^2=0 \\
\Rightarrow\;& x=2
\end{aligned}$$

Therefore, the only critical point of $V(x,y)$ inside B is the single point $(2,2)$, and thus, the maximum volume of the box is:

$$\boxed{V(2,2) = 4\text{m}^3}$$

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Exercises

1. Find the critical points of $f(x, y) = y^2 - x^2$. Then determine if each corresponds to a local minimum, local maximum, or saddle point of $f(x, y)$.
2. Find the absolute minimum and maximum values of $f(x, y) = x^2 + y^2 - 2x$ on the closed triangular region T with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$.
3. Find the shortest distance from the point $P = (1, 0, -2)$ to the plane T given by $x + 2y + z = 4$ [Hint: instead of minimizing the distance d between P and any point in T, you could minimize the square of this distance, d^2 (why is this?). This will shorten your calculations.].

Solutions

1. The only critical point of $f(x, y)$ is $(0, 0)$. It is a saddle point of $f(x, y)$.
2. The absolute maximum value of $f(x, y)$ on T is $f(0, 2) = f(0, -2) = 4$, and the absolute minimum value of $f(x, y)$ on T is $f(1, 0) = -1$. [The points you should have checked are: $(1, 0), (0, -2), (2, 0), (0, 2), (0, 0), (3/2, -1/2)$, and $(3/2, 1/2)$].
3. The minimum distance from P to T is $\frac{5}{\sqrt{6}}$. This is the distance from the point P to the point $\left(\frac{11}{6}, \frac{5}{3}, \frac{-7}{6}\right)$ in T.