14.5: The Chain Rule

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Overview

As you have seen, the rules for taking partial derivatives carry over quite nicely from single-variable calculus, and thus there is no need to devote weeks to learning new rules of differentiation as there was in single-variable calculus.

However the Chain Rule, as we currently think of it, is a bit limited. We can, of course, use it to calculate the partial derivatives of, for example, $f(x, y) = e^{xy}$. But, suppose that x and y were themselves functions of additional variables, say s and t. How could we calculate a partial derivative of f? And with respect to what variable(s) may we do so?

In this section, we will explore such problems, and expand the Chain Rule to a more general version that will better suit us in this new multivariable world.

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Single-Variable Functions

Recall the Chain Rule for single-variable functions:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(g(t)) = f'(g(t))g'(t)$$

There is another way that this rule is commonly stated: If x is the function $g(t)$, then the Chain Rule tells us how to differentiate $f(x)$ with respect to t. That is, it tells us:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x) = f'(x)x'(t)$$

Since the primes here are a bit ambiguous (as the first denotes the derivative of f with respect to x, and the second denotes the derivative of x with respect to t), this is often written:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

This latter form will connect very nicely to the expanded form of the Chain Rule we will soon introduce.

Chain Rule, Case 1

With this setup in mind, consider the following problem: Let $f(x, y) = x^2y + 3xy^4$, where $x(t) = \sin(2t)$ and $y(t) = \cos(t)$. Calculate $\frac{\mathrm{d}f}{\mathrm{d}t}$.

Note that this has the same flavor as the standard chain rule problem from single-variable calculus: we want the derivative of the function $f(x, y)$ when x and y are themselves functions of a third variable, t.

Note also that we want the *ordinary* derivative of f with respect to t, not the partial derivatives of f with respect to x and y. Why? Well, since x and y are just functions of t, f is ultimately itself a function of just one variable: t!

Chain Rule, Case 1, cont.

The most obvious way to attack this problem is to substitute $\sin(2t)$ and $\cos(t)$ in for x and y, giving:

$$f(t) = \sin^2(2t)\cos(t) + 3\sin(2t)\cos^4(t)$$

Using several Chain Rules and two product rules, we have:

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= 4\sin(2t)\cos(2t)\cos(t) - \sin^2(2t)\sin(t) + 6\cos(2t)\cos^4(t) \\ &\quad - 12\sin(2t)\cos^3(t)\sin(t)\end{aligned}$$

If you did this out by hand, you probably noticed that this problem has a *lot* of moving parts, even though $f(x, y)$ is a fairly simple function. It would be really nice to have a method that makes things quicker and more reliable by removing some of this complexity. This is the Chain Rule.

Chain Rule, Case 1, cont.

The Chain Rule, Case 1: Suppose that $f(x, y)$ is a differentiable function of x and y, and $x = x(t)$ and $y = y(t)$ are differentiable functions of t. Then f is also a differentiable function of t, with:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Note the similarity between this version of the Chain Rule and the one from single-variable calculus above!

Let's try this out on the problem above.

Example

Calculate $\frac{\mathrm{d}f}{\mathrm{d}t}$ where $f(x, y) = x^2y + 3xy^4$, $x(t) = \sin(2t)$, and $y(t) = \cos(t)$. Write your final answer in terms of the variable t.

Using the Chain Rule, we have:

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (2xy + 3y^4)(2\cos(2t)) + (x^2 + 12xy^3)(-\sin(t)) \\ &= (2\sin(2t)\cos(t) + 3\cos^4(t))(2\cos(2t)) \\ &\quad + (\sin^2(2t) + 12\sin(2t)\cos^3(t))(-\sin(t))\end{aligned}$$

Compare this with our answer above to see that we got the same thing, but with *much* less mental effort.

Further Expanding the Chain Rule

We have managed to expand the chain rule a little, but only just a little: so far we only know that we can take the ordinary derivative of a two-variable function $f(x, y)$ when x and y are themselves single-variable functions of t. This raises some key questions: what if x and y are multivariable functions? And, is there a version of the Chain Rule for functions of $f(x_1, x_2, \dots, x_n)$ of more than two variables?

We address the former first, and then the latter.

The Chain Rule, Case 2

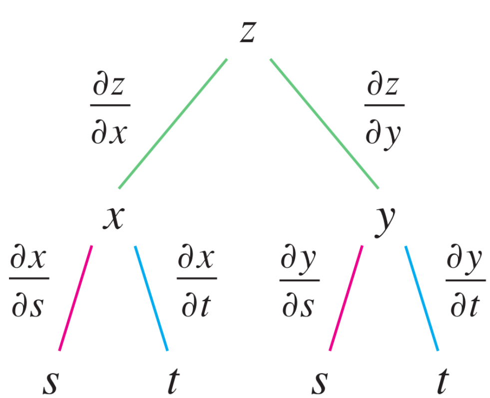
The Chain Rule, Case 2: Suppose that $f(x, y)$ is a differentiable function of x and y where $x = g(s, t)$ and $y = h(s, t)$ are themselves differentiable functions of s and t. Then:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

An analogous statement holds for $\dfrac{\partial f}{\partial t}$.

The Chain Rule, Case 2, cont.

To remember this, consider the following tree diagram:



To take the partial derivative of z with respect to, say, t, follow every path from z to t in the tree, multiplying the partial derivatives along a given path. The partial derivative is the sum all the products obtained in this way.

Example

Let $f(x, y) = e^x \sin(y)$, $x(s, t) = st^2$, and $y(s, t) = s^2 t$. Calculate $\dfrac{\partial f}{\partial s}$. Write your final answer in terms of the variables s and t.

We have:

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= (e^x \sin(y))(t^2) + (e^x \cos(y))(2st) \\ &= \boxed{t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t)}\end{aligned}$$

The General Chain Rule

There's no reason to limit f to two variables, and no need to limit those variables themselves to two variables. Thus, here is a general version of the Chain Rule:

The Chain Rule: Suppose that f is a differentiable function of the variables $x_1, x_2, \ldots, x_m$, and each x_i is itself a differentiable function of $t_1, t_2, \ldots, t_n$. Then:

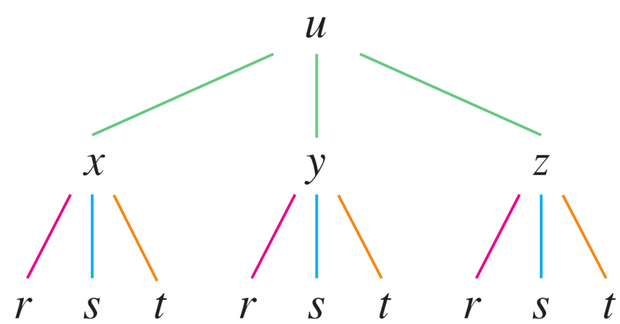
$$\frac{\partial f}{\partial t_j} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

To remember this, you can use a tree diagram in the same way as we did above. See the example below.

Example

Let $u(x, y, z) = x^4 y + y^2 z^3$, with $x(r, s, t) = rse^t$, $y(r, s, t) = rs^2 e^{-t}$, and $z(r, s, t) = r^2 s \sin(t)$. Evaluate $\frac{\partial u}{\partial s}$ at $r = 2$, $s = 1$, and $t = 0$.

We begin by drawing a tree diagram:



Example, cont.

Reading the diagram exactly as before, we have:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

giving:

$$\frac{\partial u}{\partial s} = (4x^3 y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2 z^2)(r^2 \sin(t))$$

Now, note that we have:

$$x(2, 1, 0) = 2 \qquad y(2, 1, 0) = 2 \qquad z(2, 1, 0) = 0$$

Therefore, plugging in we have:

$$\left.\frac{\partial u}{\partial s}\right|_{(r,s,t)=(2,1,0)} = (64)(2) + (16 + 0)(4) + (0)(0) = \boxed{192}$$

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Exercises

1. Let $z(x,y) = xy^3 - x^2y$, $x(t) = t^2+1$, and $y(t) = t^2-1$. Calculate $\frac{dz}{dt}$ in two different ways: by first substituting $x(t)$ and $y(t)$ into z; and second by using the Chain Rule. How do your answers compare?
2. Let $z(x,y) = (x-y)^5$, $x(s,t) = s^2t$, and $y(s,t) = st^2$. Calculate $\frac{\partial z}{\partial t}$.
3. Use a tree diagram to write out the Chain Rule for $\dfrac{\partial f}{\partial r}$, where f is a function of x and y; $x = x(r,s,t)$; and $y = y(r,s,t)$.
4. Let $z = x^4 + x^2y$, $x = s+2t-u$, and $y = stu^2$. Calculate $\dfrac{\partial z}{\partial s}$ when $s = 4$, $t = 2$, and $u = 1$.

Solutions

1. Either method should yield:

$$\frac{dz}{dt} = (2t)((t^2-1)^3 - 2(t^2+1)(t^2-1) + 3(t^2+1)(t^2-1)^2 - (t^2+1)^2)$$

2. $\dfrac{\partial z}{\partial t} = 5(s^2t - st^2)^4(s^2 - 2st)$
3. The tree is left to you (though you can certainly check with me to verify your result). The Chain Rule is:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

4. $\left.\dfrac{\partial z}{\partial s}\right|_{(s,t,u)=(4,2,1)} = 1582$