

14.4: Tangent Planes and Linear Approximations

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Overview

In the previous section, we learned to calculate the instantaneous rate of change of a function $f(x, y)$ at a point (a, b) in exactly two directions: parallel to the x -axis and parallel to the y -axis. Believe it or not, just these two directions are enough to get us to our first application of derivatives: tangent planes.

Recall that in single-variable calculus, you can use the derivative of a function $f(x)$ at a point to give an equation of the tangent line to f at that point. Given a two-variable function $f(x, y)$, the partial derivatives at a point can be used to specify a similar object: a plane tangent to the graph of f . In this section we will discuss how to construct such a tangent plane, and then learn how to give an equation for it. We will then turn to one of its uses: estimating values of $f(x, y)$.

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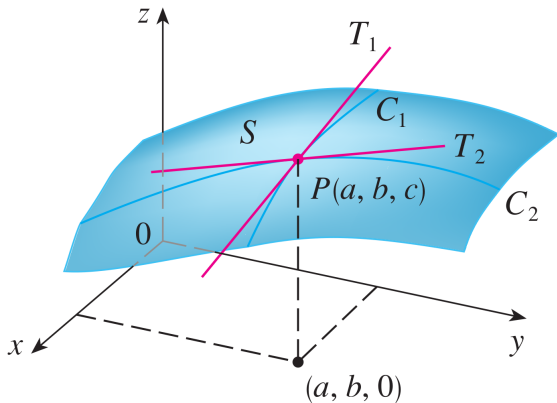
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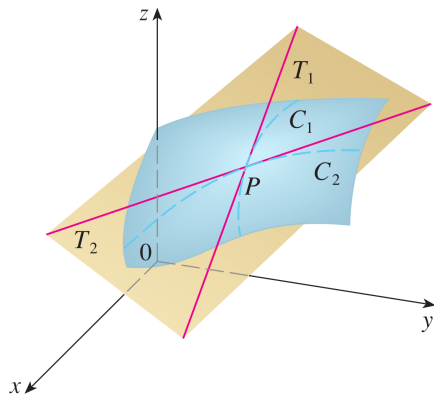
What is the Tangent Plane?

Suppose we have a two-variable function $f(x, y)$ whose graph is the surface S . Recall from the previous section that the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ of f give the respective slopes of the lines T_1 and T_2 that lie tangent to S at the point $P = (a, b, c)$ as in the following figure:



What is the Tangent Plane?, cont.

Note that the lines T_1 and T_2 generate a unique plane that contains them both:



This is the plane tangent to S at the point P , i.e., the tangent plane at P , so called because it contains the two tangent lines. Note that it, too, lies tangent to S .

Toward an Equation

This is a nice definition, but it tells us very little about how to give an equation for such a plane. That is our next goal.

Recall the generic vector equation for a plane in \mathbb{R}^3 :

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

where \vec{n} is a vector orthogonal (essentially, perpendicular) to the plane; \vec{r} is the vector $\langle x, y, z \rangle$ whose tail is placed at the origin and whose head is a generic point in the plane; and \vec{r}_0 is a vector whose tail is placed at the origin and whose head is a known point on the plane.

In order to give an equation for the tangent plane on the previous slides, we need to find suitable vectors to serve as \vec{n} and \vec{r}_0 .

Finding \vec{r}_0

Let's begin with \vec{r}_0 .

Notice that the tangent lines T_1 and T_2 pass through the point P on the graph of $f(x, y)$. Therefore the tangent plane, which contains both tangent lines, does, too. To work out the vector \vec{r}_0 , then, we just need to know the coordinates of P .

Recall that the graph of $f(x, y)$ is the set of all points (x, y, z) in \mathbb{R}^3 satisfying $z = f(x, y)$, where (x, y) is in the domain of f . Therefore, we may construct the graph of $f(x, y)$ point-by-point by choosing a point (x_0, y_0) in the domain of $f(x, y)$, plugging (x_0, y_0) into $f(x, y)$, and then plotting the resulting point $(x_0, y_0, f(x_0, y_0))$ on the graph of $f(x, y)$.

Finding \vec{r}_0 , cont.

We will assume, following our sketch on a previous slide, that the x - and y -coordinates of P are a and b , respectively. Therefore, by the construction on the previous slide, P has coordinates $(a, b, f(a, b))$, so that we may write:

$$\vec{r}_0 := \langle a, b, f(a, b) \rangle$$

A Normal Vector

Now we will find a normal vector \vec{n} .

Since a normal vector is orthogonal to the tangent plane, it must also be orthogonal to the tangent lines T_1 and T_2 , as these lie in the tangent plane. This observation will help to simplify our efforts.

How can we find a vector orthogonal to these two intersecting lines? Here's the key idea: Both lines contain direction vectors. Therefore, if we find a direction vector for each line, we can find a vector perpendicular to both of these (and hence both tangent lines, and therefore the tangent plane) using the cross product!

Let's get to it.

A Normal Vector, cont.

We begin by finding a direction vector for the line T_1 .

Recall that the slope of T_1 is $f_x(a, b)$. That is, for any two points (x_1, b, z_1) and (x_2, b, z_2) on T_1 , we have:

$$\frac{(z_2 - z_1)}{(x_2 - x_1)} = \frac{\Delta z}{\Delta x} = f_x(a, b) = \frac{f_x(a, b)}{1}$$

In particular, if we choose x_1 and x_2 so that they are one unit apart, the constant ratio above tells us that $z_2 - z_1 = f_x(a, b)$, and hence the vector connecting these two points on T_1 is:

$$\vec{v}_1 := \langle x_2 - x_1, b - b, z_2 - z_1 \rangle = \langle 1, 0, f_x(a, b) \rangle$$

This is a direction vector for T_1 .

A Normal Vector, cont.

A similar argument to the one on the previous slide tells us that a direction vector for T_2 is:

$$\vec{v}_2 := \langle 0, 1, f_y(a, b) \rangle$$

(convince yourself of this).

Therefore, a vector perpendicular to both, and hence a vector orthogonal to the tangent plane, is:

$$\vec{n} := \vec{v}_2 \times \vec{v}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & f_y(a, b) \\ 1 & 0 & f_x(a, b) \end{vmatrix} = \langle f_x(a, b), f_y(a, b), -1 \rangle$$

An Equation

Therefore, putting everything together, an equation of the plane tangent to the graph of $f(x, y)$ at the point $(a, b, f(a, b))$ is:

$$\langle f_x(a, b), f_y(a, b), -1 \rangle \cdot (\langle x, y, z \rangle - \langle a, b, f(a, b) \rangle) = 0$$

which, in scalar form is:

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$$

Most commonly, this is rearranged to:

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

Example

Find an equation of the plane T_0 that lies tangent to the surface $2x^2 + y^2 - z = 0$ at the point $(1, 3, 11)$.

On the previous slide, we gave a generic equation for the plane that lies tangent to the graph of a function $f(x, y)$ at a given point. To utilise that formula here, we must first work out what function has the surface $2x^2 + y^2 - z = 0$ as its graph.

Note that we may rearrange the equation $2x^2 + y^2 - z = 0$ as follows:

$$z = 2x^2 + y^2$$

Then, recall again that the graph of the function $f(x, y)$ consists of all points (x, y, z) satisfying $z = f(x, y)$. Therefore, the function

$$g(x, y) := 2x^2 + y^2$$

has the surface $z = 2x^2 + y^2$ (i.e. $2x^2 + y^2 - z = 0$) as its graph.

Example, cont.

From the statement of the problem and our work on the previous two slides, we know that an equation for T_0 will have the general form:

$$z = g_x(1, 3)(x - 1) + g_y(1, 3)(y - 3) + 11$$

Let's calculate the partial derivatives:

$$g_x(x, y) = 4x \Rightarrow g_x(1, 3) = 4$$

$$g_y(x, y) = 2y \Rightarrow g_y(1, 3) = 6$$

Therefore, putting all of this together, an equation for T_0 is:

$$z = 4(x - 1) + 6(y - 3) + 11$$

Or, in linear form:

$$z = 4x + 6y - 11$$

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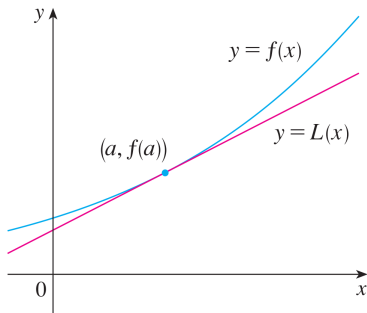
Exercises

Framing

We went to all this trouble to define the tangent plane and work out an equation for it, so a question now confronts us: what can we use this for? We turn back to single-variable calculus for inspiration.

Single-Variable Calculus

Recall that the derivative of a function $f(x)$ at $x = a$ can be used to give an equation for the line $L(x)$ that lies tangent to the graph of $f(x)$ at the point $(a, f(a))$:



Note in particular that the values of $L(x)$ are close to the values of $f(x)$ when x is near a , so, the values of $L(x)$ can be used to approximate the values of $f(x)$ near $x = a$.

Linear Approximation

In many cases, this observation can help us save time and energy. Suppose f is a computationally expensive function, like, say:

$$f(x) = \ln(\cos(\pi(x+6))) - |x-4|$$

Suppose we want to know a value of $f(x)$ near $x=4$; say e.g. $f(4.1)$. $f(4)$ is relatively easy to compute (try it!), but $f(4.1)$ is decidedly not. Since $f'(4)$ is also relatively straightforward to compute, depending on the level of accuracy needed it may be worth it to instead approximate $f(4.1)$ with the computationally inexpensive tangent line at $x=4$:

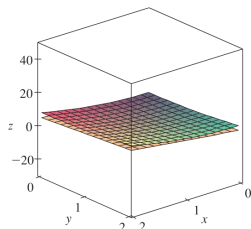
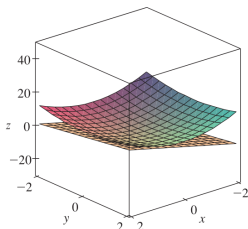
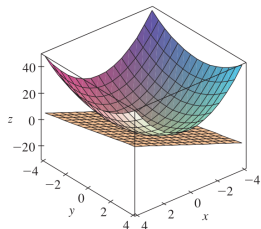
$$L(x) = 4 - x$$

as $f(4.1) \approx L(4.1)$, and $4 - 4.1 = -0.1$ is much easier to calculate (by hand or machine) than $f(4.1)$.

The Picture

In \mathbb{R}^3 we have an analogous picture.

Below are three images of a surface and the plane tangent to that surface at a point. From left to right, we gradually zoom in on the point where the two meet:



The Picture, cont.

As we zoom in, the plane and the surface become almost indistinguishable from one another. Thus, if this surface is the graph of a two-variable function $f(x, y)$, we can use the tangent plane to estimate values of f near the point of intersection.

Linear Approximation

Recall that an equation of the plane tangent to the graph of $f(x, y)$ at $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

Since the points on the plane are close to the points on the graph of $z = f(x, y)$ when (x, y) is near (a, b) , we have:

$$f(x, y) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

when (x, y) is near (a, b) . This entire expression is called the **linear approximation** or **tangent plane approximation** of f at (a, b) . The right-hand side alone is called the **linearization** of f at (a, b) , often written:

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

Example

We saw above that an equation of the plane that lies tangent to the graph of $g(x, y) = 2x^2 + y^2$ at the point $(1, 3, 11)$ is $z = 4x + 6y - 11$. Use this to estimate $g(1.1, 2.9)$.

From the previous slide, we have $g(x, y) \approx 4x + 6y - 11$. Therefore:

$$g(1.1, 2.9) \approx 4(1.1) + 6(2.9) - 11 = 10.8$$

Example

Find the linearization $L(x, y)$ of $f(x, y) = xe^{xy}$ at $(1, 0)$ and use it to approximate $f(1.1, -0.1)$.

Recall that the linearization of f at $(1, 0)$ is simply the right side of an equation of the plane tangent to f at $(1, 0)$. So, we begin by finding the latter. An equation for the plane that lies tangent to the graph of $f(x, y)$ at the point $(1, 0)$ is:

$$z = f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) + f(1, 0)$$

We have:

$$f_x(x, y) = e^{xy} + xye^{xy} \Rightarrow f_x(1, 0) = 1$$

$$f_y(x, y) = x^2 e^{xy} \Rightarrow f_y(1, 0) = 1$$

$$f(1, 0) = 1$$

Example, cont.

Therefore, an equation of the plane that lies tangent to the graph of $f(x, y)$ at the point $(1, 0, 1)$ is:

$$z = 1(x - 1) + 1(y - 0) + 1$$

Thus, the linearization of f at $(1, 0)$ is

$$L(x, y) = (x - 1) + y + 1$$

which gives:

$$f(1.1, -0.1) \approx L(1.1, -0.1) = (1.1 - 1) - 0.1 + 1 = 1$$

A Final Note

You can also create linear approximations for functions of more variables, and the equation is wholly analogous. For example, for a function of three variables, we can approximate it near (a, b, c) using:

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

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1. Find an equation of the plane T that lies tangent to the graph of $z = (x + 2)^2 - 2(y - 1)^2 - 5$ at $(2, 3, 3)$.
2. Find the linearization $L(x, y)$ of $f(x, y) = \sqrt{xy}$ at $(1, 4)$.
3. Use the linearization you found in the previous exercise to estimate $f(1.1, 3.9)$.

Solutions

1. One possible equation for T is $z = 8x - 8y + 11$.
2. $L(x, y) = x + \frac{y}{4}$
3. $f(1.1, 3.9) \approx L(1.1, 3.9) = 2.075$