14.4: Tangent Planes and Linear Approximations

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Overview

In the previous section, we learned to calculate the instantaneous rate of change of a function $f(x, y)$ at a point (a, b) in exactly two directions: parallel to the x-axis and parallel to the y-axis. Believe it or not, just these two directions are enough to get us to our first application of derivatives: tangent planes.

Recall that in single-variable calculus, you can use the derivative of a function $f(x)$ at a point to give an equation of the tangent line to f at that point. Given a two-variable function $f(x, y)$, the partial derivatives at a point can be used to specify a similar object: a plane tangent to the graph of f. In this section we will discuss how to construct such a tangent plane, and then learn how to give an equation for it. We will then turn to one of its uses: estimating values of $f(x, y)$.

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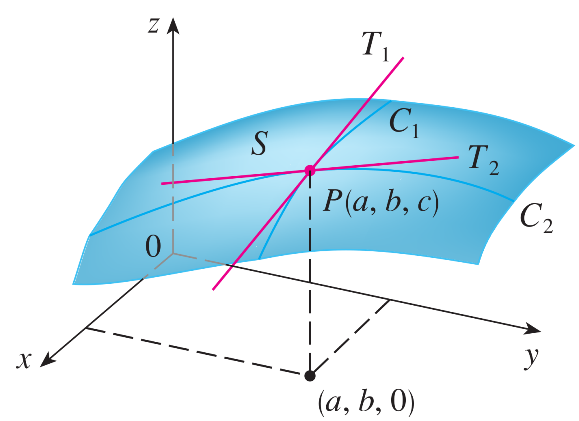
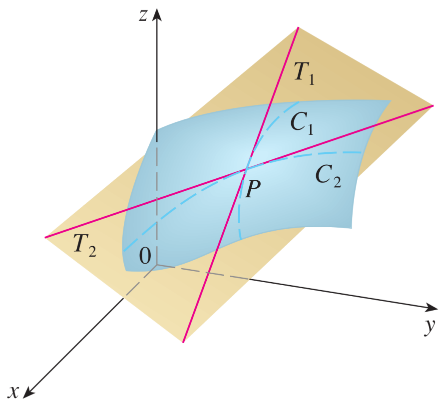
What is the Tangent Plane?

Suppose we have a two-variable function $f(x, y)$ whose graph is the surface S. Recall from the previous section that the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ of f give the respective slopes of the lines T_1 and T_2 that lie tangent to S at the point $P = (a, b, c)$ as in the following figure:



What is the Tangent Plane?, cont.

Note that the lines T_1 and T_2 generate a unique plane that contains them both:



This is the plane tangent to S at the point P, i.e., the tangent plane at P, so called because it contains the two tangent lines. Note that it, too lies tangent to S.

Toward an Equation

This is a nice definition, but it tells us very little about how to give an equation for such a plane. That is our next goal.

Recall the generic vector equation for a plane in $\mathbb{R}^3$:

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

where $\vec{n}$ is a vector orthogonal (essentially, perpendicular) to the plane; $\vec{r}$ is the vector $\langle x, y, z \rangle$ whose tail is placed at the origin and whose head is a generic point in the plane; and $\vec{r}_0$ is a vector whose tail is placed at the origin and whose head is a known point on the plane.

In order to give an equation for the tangent plane on the previous slides, we need to find suitable vectors to serve as $\vec{n}$ and $\vec{r_0}$.

Finding $\vec{r_0}$

Let's begin with $\vec{r_0}$.

Notice that the tangent lines T_1 and T_2 pass through the point P on the graph of $f(x, y)$. Therefore the tangent plane, which contains both tangent lines, does, too. To work out the vector $\vec{r_0}$, then, we just need to know the coordinates of P.

Recall that the graph of $f(x, y)$ is the set of all points (x, y, z) in $\mathbb{R}^3$ satisfying $z = f(x, y)$, where (x, y) is in the domain of f. Therefore, we may construct the graph of $f(x, y)$ point-by-point by choosing a point (x_0, y_0) in the domain of $f(x, y)$, plugging (x_0, y_0) into $f(x, y)$, and then plotting the resulting point $(x_0, y_0, f(x_0, y_0))$ on the graph of $f(x, y)$.

Finding $\vec{r_0}$, cont.

We will assume, following our sketch on a previous slide, that the x- and y-coordinates of P are a and b, respectively. Therefore, by the construction on the previous slide, P has coordinates $(a, b, f(a, b))$, so that we may write:

$$\vec{r_0} := \big\langle a, b, f(a, b) \big\rangle$$

A Normal Vector

Now we will find a normal vector $\vec{n}$.

Since a normal vector is orthogonal to the tangent plane, it must also be othogonal to the tangent lines T_1 and T_2, as these lie in the tangent plane. This observation will help to simplify our efforts.

How can we find a vector orthogonal to these two intersecting lines? Here's the key idea: Both lines contain direction vectors. Therefore, if we find a direction vector for each line, we can find a vector perpendicular to both of these (and hence both tangent lines, and therefore the tangent plane) using the cross product!

Let's get to it.

A Normal Vector, cont.

We begin by finding a direction vector for the line T_1.

Recall that the slope of T_1 is $f_x(a, b)$. That is, for any two points (x_1, b, z_1) and (x_2, b, z_2) on T_1, we have:

$$\frac{(z_2 - z_1)}{(x_2 - x_1)} = \frac{\Delta z}{\Delta x} = f_x(a, b) = \frac{f_x(a, b)}{1}$$

In particular, if we choose x_1 and x_2 so that they are one unit apart, the constant ratio above tells us that $z_2 - z_1 = f_x(a, b)$, and hence the vector connecting these two points on T_1 is:

$$\vec{v_1} := \langle x_2 - x_1, b - b, z_2 - z_1 \rangle = \big\langle 1, 0, f_x(a, b) \big\rangle$$

This is a direction vector for T_1.

A Normal Vector, cont.

A similar argument to the one on the previous slide tells us that a direction vector for T_2 is:

$$\vec{v_2} := \big\langle 0, 1, f_y(a, b) \big\rangle$$

(convince yourself of this).

Therefore, a vector perpendicular to both, and hence a vector orthogonal to the tangent plane, is:

$$\vec{n} := \vec{v_2} \times \vec{v_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & f_y(a, b) \\ 1 & 0 & f_x(a, b) \end{vmatrix} = \big\langle f_x(a, b), f_y(a, b), -1 \big\rangle$$

An Equation

Therefore, putting everything together, an equation of the plane tangent to the graph of $f(x, y)$ at the point $(a, b, f(a, b))$ is:

$$\big\langle f_x(a, b), f_y(a, b), -1 \big\rangle \cdot \Big(\langle x, y, z \rangle - \big\langle a, b, f(a, b) \big\rangle \Big) = 0$$

which, in scalar form is:

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$$

Most commonly, this is rearranged to:

$$\boxed{z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)}$$

Example

Find an equation of the plane T_0 that lies tangent to the surface $2x^2 + y^2 - z = 0$ at the point $(1, 3, 11)$.

On the previous slide, we gave a generic equation for the plane that lies tangent to the graph of a function $f(x, y)$ at a given point. To utilise that formula here, we must first work out what function has the surface $2x^2 + y^2 - z = 0$ as its graph.

Note that we may rearrange the equation $2x^2 + y^2 - z = 0$ as follows:

$$z = 2x^2 + y^2$$

Then, recall again that the graph of the function $f(x, y)$ consists of all points (x, y, z) satisfying $z = f(x, y)$. Therefore, the function

$$g(x, y) := 2x^2 + y^2$$

has the surface $z = 2x^2 + y^2$ (i.e. $2x^2 + y^2 - z = 0$) as its graph.

Example, cont.

From the statement of the problem and our work on the previous two slides, we know that an equation for T_0 will have the general form:

$$z = g_x(1, 3)(x - 1) + g_y(1, 3)(y - 3) + 11$$

Let's calculate the partial derivatives:

$$g_x(x, y) = 4x \quad \Rightarrow \quad g_x(1, 3) = 4$$
$$g_y(x, y) = 2y \quad \Rightarrow \quad g_y(1, 3) = 6$$

Therefore, putting all of this together, an equation for T_0 is:

$$\boxed{z = 4(x - 1) + 6(y - 3) + 11}$$

Or, in linear form:

$$\boxed{z = 4x + 6y - 11}$$

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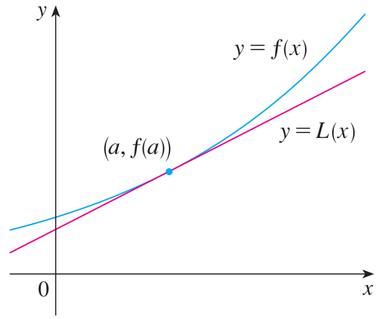
Framing

We went to all this trouble to define the tangent plane and work out an equation for it, so a question now confronts us: what can we use this for? We turn back to single-variable calculus for inspiration.

Single-Variable Calculus

Recall that the derivative of a function $f(x)$ at $x = a$ can be used to give an equation for the line $L(x)$ that lies tangent to the graph of $f(x)$ at the point $(a, f(a))$:



Note in particular that the values of $L(x)$ are close to the values of $f(x)$ when x is near a, so, the values of $L(x)$ can be used to approximate the values of $f(x)$ near $x = a$.

Linear Approximation

In many cases, this observation can help us save time and energy. Suppose f is a computationally expensive function, like, say:

$$f(x) = \ln\left(\cos(\pi(x+6)) - |x-4|\right)$$

Suppose we want to know a value of $f(x)$ near $x = 4$; say e.g. $f(4.1)$. $f(4)$ is relatively easy to compute (try it!), but $f(4.1)$ is decidedly not. Since $f'(4)$ is also relatively straightforward to compute, depending on the level of accuracy needed it may be worth it to instead approximate $f(4.1)$ with the computationally inexpensive tangent line at $x = 4$:

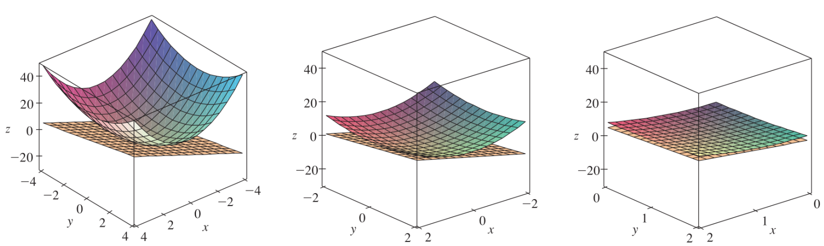
$$L(x) = 4 - x$$

as $f(4.1) \approx L(4.1)$, and $4 - 4.1 = -0.1$ is much easier to calculate (by hand or machine) than $f(4.1)$.

The Picture

In $\mathbb{R}^3$ we have an analogous picture.

Below are three images of a surface and the plane tangent to that surface at a point. From left to right, we gradually zoom in on the point where the two meet:



The Picture, cont.

As we zoom in, the plane and the surface become almost indistinguishable from one another. Thus, if this surface is the graph of a two-variable function $f(x, y)$, we can use the tangent plane to estimate values of f near the point of intersection.

Linear Approximation

Recall that an equation of the plane tangent to the graph of $f(x,y)$ at $(a,b,f(a,b))$ is

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Since the points on the plane are close to the points on the graph of $z = f(x,y)$ when (x,y) is near (a,b), we have:

$$f(x,y) \approx f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

when (x,y) is near (a,b). This entire expression is called the **linear approximation** or **tangent plane approximation** of f at (a,b). The right-hand side alone is called the **linearization** of f at (a,b), often written:

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Example

We saw above that an equation of the plane that lies tangent to the graph of $g(x,y) = 2x^2 + y^2$ at the point $(1,3,11)$ is $z = 4x + 6y - 11$. Use this to estimate $g(1.1, 2.9)$.

From the previous slide, we have $g(x,y) \approx 4x + 6y - 11$. Therefore:

$$\boxed{g(1.1, 2.9) \approx 4(1.1) + 6(2.9) - 11 = 10.8}$$

Example

Find the linearization $L(x,y)$ of $f(x,y) = xe^{xy}$ at $(1,0)$ and use it to approximate $f(1.1, -0.1)$.

Recall that the linearization of f at $(1,0)$ is simply the right side of an equation of the plane tangent to f at $(1,0)$. So, we begin by finding the latter. An equation for the plane that lies tangent to the graph of $f(x,y)$ at the point $(1,0)$ is:

$$z = f_x(1,0)(x-1) + f_y(1,0)(y-0) + f(1,0)$$

We have:

$$f_x(x,y) = e^{xy} + xye^{xy} \;\Rightarrow\; f_x(1,0) = 1$$

$$f_y(x,y) = x^2e^{xy} \;\Rightarrow\; f_y(1,0) = 1$$

$$f(1,0) = 1$$

Example, cont.

Therefore, an equation of the plane that lies tangent to the graph of $f(x,y)$ at the point $(1,0,1)$ is:

$$z = 1(x-1) + 1(y-0) + 1$$

Thus, the linearization of f at $(1,0)$ is

$$\boxed{L(x,y) = (x-1) + y + 1}$$

which gives:

$$\boxed{f(1.1, -0.1) \approx L(1.1, -0.1) = (1.1-1) - 0.1 + 1 = 1}$$

A Final Note

You can also create linear approximations for functions of more variables, and the equation is wholly analogous. For example, for a function of three variables, we can approximate it near (a, b, c) using:

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

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Exercises

1. Find an equation of the plane T that lies tangent to the graph of $z = (x + 2)^2 - 2(y - 1)^2 - 5$ at $(2, 3, 3)$.
2. Find the linearization $L(x, y)$ of $f(x, y) = \sqrt{xy}$ at $(1, 4)$.
3. Use the linearization you found in the previous exercise to estimate $f(1.1, 3.9)$.

Solutions

1. One possible equation for T is $z = 8x - 8y + 11$.
2. $L(x, y) = x + \frac{y}{4}$
3. $f(1.1, 3.9) \approx L(1.1, 3.9) = 2.075$