14.3: Partial Derivatives

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Overview

At last, we arrive at differentiation. In this section we begin by learning how to take derivatives of two-variable functions, how to denote these derivatives, and how to interpret them graphically. We'll also apply our methods to computing derivatives of functions of more than two variables. These tasks completed, we will then examine how other core derivative concepts from single-variable calculus apply here, namely: implicit differentiation and higher-order derivatives.

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Single-Variable Functions

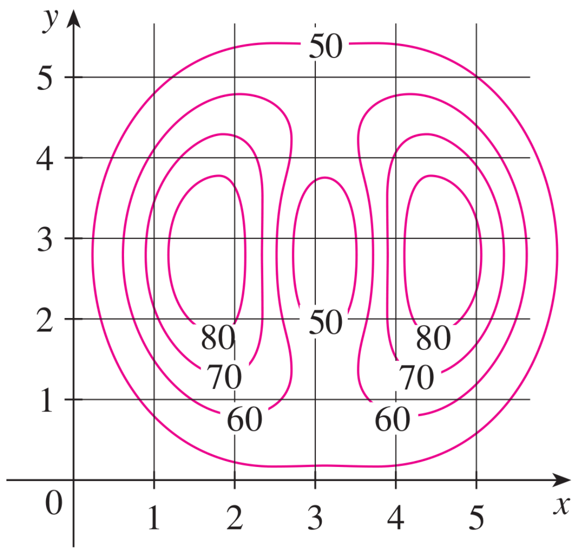
Recall that the derivative of a single-variable function $f(x)$ at $x = a$ is defined as follows:

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

This definition is designed specifically to tell us the instantaneous rate of change of f at $x = a$.

With this motivation in mind, how might we define the derivative of, say, a two-variable function $f(x, y)$? Could we design it to give us the instantaneous rate of change of $f(x, y)$ at a point, (a, b)? Let's think about this idea a bit more carefully with the contour map of a sample function $f(x, y)$ in front of us.

A Two-Variable Function



The Derivative?

Now suppose we wanted to calculate the instantaneous rate of change of f at the point $(2, 1)$, something we would like to call $f'(2, 1)$. Unfortunately, this is inherently ambiguous. Indeed, if we move vertically along the line $x = 2$, we see that $f'(2, 1)$ should be a fairly large positive number, as f rapidly incresases to 70, and then 80 before we make it to $(2, 2)$. On the other hand, if we move horizontally along $y = 1$, the value of f actually seems to *decrease* very slightly, suggesting a very small negative value for $f'(2, 1)$. Every possible direction we might head away from $(2, 1)$ in gives a different instantaneous rate of change of f!

Thus, there is no way to define "the" derivative of a two-variable function at a point!

Not "the" Derivative, but Derivatives!

Instead, to give an unambiguous derivative (i.e. the instantaneous rate of change of a function) we need to specify two things: the point at which we wish to take such a derivative, and the direction in which we wish to take it. Today — for two-variable functions, anyway — we will only talk about two directions: parallel to the x-axis, and parallel to the y-axis. These are the so-called **partial derivatives**.

Definition for Two-Variable Functions

The **partial derivative of** f **with respect to** x **at** (a, b) is:

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

This tells us the instantaneous rate at which f is changing at (a, b) when we move parallel to the x-axis in the direction of increasing x, with y held fixed.

Similarly the **partial derivative of** f **with respect to** y **at** (a, b) is:

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

This tells us the instantaneous rate at which f is changing at (a, b) when we move parallel to the y-axis in the direction of increasing y, with x held fixed.

Contour Map, Revisited

If we examine the contour map on the slide above, then, as we discussed, we see that $f_x(2,1)$ is a negative number near zero, and $f_y(2,1)$ is a comparatively large positive number.

Calculation

We now know how the two partial derivatives of a two-variable function are defined and what they represent verbally. We certainly don't want to do calculations with the limit definition of the derivative, as this is just as labor intensive as it was for single-variable functions. We would like a quick way of computing partial derivatives. Here it is:

To calculate $f_x(x,y)$, think of y as a constant and differentiate $f(x,y)$ with respect to x. Calculate $f_y(x,y)$ similarly.

Why can we do this? Look back to the definition of, e.g., $f_x(x,y)$. In this definition, y is held constant. Therefore, we can treat it like one in our calculations. A similar argument holds for $f_y(x,y)$.

Example

Let $f(x,y) = x^3 + x^2y^3 - 2y^2$. Calculate $f_x(2,1)$ and $f_y(2,1)$.

From the previous slide, we have:

$$f_x(x,y) = 3x^2 + 2xy^3$$

and

$$f_y(x,y) = 3x^2y^2 - 4y$$

Thus:

$$\boxed{f_x(2,1) = 12 + 4 = 16}$$

$$\boxed{f_y(2,1) = 12 - 4 = 8}$$

A Note on Notation

There are several ways to denote partial derivatives of a function $z = f(x,y)$:

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = \frac{\partial z}{\partial x}$$

Similar notation holds for $f_y(x,y)$.

So, for example, in the exercise we just completed we could have written

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^3$$

and

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(2,1)} = 16$$

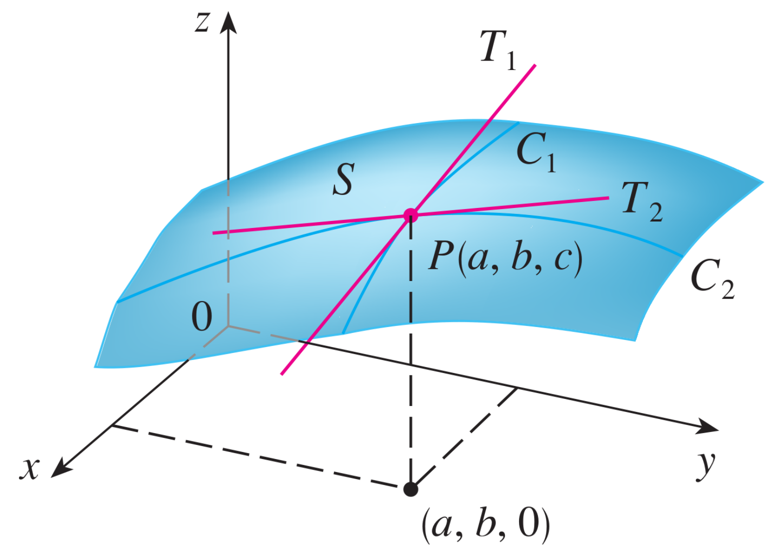
Graphical Interpretation

For a single-variable function $f(x)$, $f'(a)$ represents the slope of the line tangent to the graph of f at $(a, f(a))$. With that in mind, how might we graphically interpret $f_x(a, b)$ and $f_y(a, b)$?

Let's start with $f_x(a, b)$. This is the instantaneous rate of change of f at (a, b) when we move parallel to the x-axis, i.e., when $y = b$ is held constant. How does this play out graphically?

When we hold $y = b$ constant, we obtain a cross-sectional curve C_1 from the graph of $f(x, y)$, which runs parallel to the x-axis (see the following slide and the board for a picture of this scenario). If we let T_1 be the line tangent to C_1 at the point (a, b, c), then $f_x(a, b)$ represents the slope of that line (slope, in this case, being the change in z over the change in x).

Graphical Interpretation, cont.



Graphical Interpretation, cont.

Put more technically, if we let $g(x) = f(x, b)$ be a single-variable function, then C_1 is the graph of g, and $f_x(a, b) = g'(a)$ is the slope of the tangent line T_1 to C_1.

Similarly, if we hold $x = a$ constant, $f(a, y)$ sweeps out a curve C_2 in S which runs parallel to the y-axis. $f_y(a, b)$ is the slope of the line T_2 tangent to C_2 at (a, b, c) (slope being change in z over change in y).

We will return to this interpretation in the following section, when we talk about tangent planes.

Implicit Functions: A Reminder

Recall that when an equation involves two variables, say x and y, we can think of y as being an implicit "function" of x, because its value depends indirectly on the value of x. For example, consider the equation:

$$x^2 + y^2 = 1$$

Choosing a value for x narrows the values for y substantially. For example, if we let $x = 1$, then for the equation to be true, y *must* be 0. If we choose $x = 0$, y must be ± 1.

Implicit Differentiation

If we have an equation that makes, say, y an implicit function of x, then we can differentiate that equation by acting as though y is a true function of x and applying derivative rules accordingly. For example, to calculate $\frac{dy}{dx}$ when $x^2 + y^2 = 1$:

$$2x + 2y\frac{dy}{dx} = 0$$

by the chain rule, so that

$$\frac{dy}{dx} = \frac{-x}{y}$$

We can use these same techniques when working with an equation involving *three* variables and taking, say, z to be a function of x and y, as illustrated in the following example.

Example

Given the equation:

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Calculate $\dfrac{\partial z}{\partial x}$.

Since we do not have an explicit relationship between x, y, and z of the form $z = f(x, y)$, we *must* use implicit differentiation. We begin by computing the partial derivative of both sides of the above equation with respect to x. To do so, we must both treat y as a constant and z as a function of x and y. Differentiating, we obtain:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Solving for $\frac{\partial z}{\partial x}$ we have:

$$\boxed{\frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy}}$$

More Variables

To compute partial derivatives when there are more variables, the process is essentially the same: treat all variables as constants except the one we are differentiating with respect to, and then differentiate as you normally would with single-variable functions.

Example

Calculate $f_y(x, y, z)$ for $f(x, y, z) = e^{xy}\ln(z)$.

Regarding x and z as constants, we have:

$$\boxed{f_y(x, y, z) = xe^{xy}\ln(z)}$$

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Second-Order Partial Derivatives

Suppose that we have a function $f(x, y)$ (though the number of variables is irrelevant). Its derivative $f_x(x, y)$ is also a function of two variables, so we can often take its partial derivatives, too! Here's how we denote f's second-order partial derivatives:

$$\frac{\partial}{\partial x} f_x(x,y) = (f_x)_x(x,y) = f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$
$$\frac{\partial}{\partial y} f_x(x,y) = (f_x)_y(x,y) = f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$
$$\frac{\partial}{\partial x} f_y(x,y) = (f_y)_x(x,y) = f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$
$$\frac{\partial}{\partial y} f_y(x,y) = (f_y)_y(x,y) = f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Example

Calculate all of the second-order partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

First, we have:

$$f_x(x,y) = 3x^2 + 2xy^3$$

and

$$f_y(x,y) = 3x^2y^2 - 4y$$

Therefore, the second-order partial derivatives of $f(x, y)$ are:

$$\boxed{\begin{aligned} f_{xx}(x,y) &= 6x + 2y^3 \\ f_{xy}(x,y) &= 6xy^2 \\ f_{yx}(x,y) &= 6xy^2 \\ f_{yy}(x,y) &= 6x^2y - 4 \end{aligned}}$$

Clairaut's Theorem

In the previous example, the mixed partial derivatives f_{xy} and f_{yx} were the same. Interestingly, this is no accident, and happens quite often.

Clairaut's Theorem: If f is defined on a disk D that contains the point (a, b), and the mixed partial derivatives f_{xy} and f_{yx} are continuous on D, then:

$$f_{xy}(a,b) = f_{yx}(a,b)$$

This is technical, but it essentially says that if the mixed partial derivatives are continuous, then they are identical.

Even Higher Derivatives

We can, of course, continue taking partial derivatives of our functions, and we can also consider higher derivatives when there are more variables:

Example

Calculate $f_{xxyz}(x, y, z)$ for $f(x, y, z) = \sin(3x + yz)$.

We have:

$$\begin{aligned} & f_x(x,y,z) = 3\cos(3x+yz) \\ \Rightarrow \quad & f_{xx}(x,y,z) = -9\sin(3x+yz) \\ \Rightarrow \quad & f_{xxy}(x,y,z) = -9z\cos(3x+yz) \\ \Rightarrow \quad & \boxed{f_{xxyz}(x,y,z) = -9\cos(3x+yz) + 9yz\sin(3x+yz)} \end{aligned}$$

Clairaut's Theorem, Revisited

By the way, Clairaut's Theorem applies to these higher derivatives, too. For example, if f_{xxy}, f_{xyx}, and f_{yxx} are all continuous, then they are all equal. A similar statement holds for, e.g. f_{xyz}, f_{xzy}, f_{yxz}, f_{yzx}, f_{zxy}, and f_{zyx}. This can save you a lot of calculation time!

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Exercises

1. Let $f(x, y, z) = e^{xy} \ln(z)$. Complete the example in the slides by calculating $f_x(x, y, z)$ and $f_z(x, y, z)$.
2. Let $f(x, y) = y \arcsin(xy)$. Calculate $f_y(1, 1/2)$.
3. Let $x^2 - y^2 + z^2 - 2z = 4$. Use implicit differentiation to calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
4. Calculate all the second-order partial derivatives of $f(x, y) = \ln(2x + 3y)$. Does Clairaut's theorem hold?
5. Let $f(x, y) = \sin(2x + 5y)$. Calculate $f_{yxy}(x, y)$.
6. Recall that the derivative $f_{xy}(x, y)$ may be written in Leibniz notation as $\frac{\partial^2 f}{\partial y \partial x}$. How do we write $f_{xxyx}(x, y)$ in Leibniz notation?

Solutions

1. $f_x(x, y, z) = ye^{xy} \ln(z)$; $f_z(x, y) = \frac{e^{xy}}{z}$.
2. $f_y(1, 1/2) = \frac{\pi + 2\sqrt{3}}{6}$, since $f_y(x, y) = \arcsin(xy) + \frac{xy}{\sqrt{1-(xy)^2}}$.
3. $\frac{\partial z}{\partial x} = \frac{x}{1-z}$ and $\frac{\partial z}{\partial y} = \frac{y}{z-1}$.
4. $f_{xx}(x, y) = \frac{-4}{(2x+3y)^2}$, $f_{xy}(x, y) = \frac{-6}{(2x+3y)^2}$, $f_{yx}(x, y) = \frac{-6}{(2x+3y)^2}$, and $f_{yy}(x, y) = \frac{-9}{(2x+3y)^2}$. Clairaut's theorem does, indeed, hold, as $f_{xy}(x, y) = f_{yx}(x, y)$.
5. $f_{yxy}(x, y) = -50 \cos(2x + 5y)$.
6. $f_{xxyx}(x, y)$ is $\frac{\partial^4 f}{\partial x \partial y \partial x^2}$ in Leibniz notation.