14.2: Limits and Continuity

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Section Overview

Our first goal in this chapter is to calculate the derivative of a multivariable function. Recall from single-variable calculus that the derivative of a single-variable function $f(x)$ at $x = a$ is defined as a limit:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Our definition for multivariable functions will also involve a limit, which means we will first need to make sense of limits of multivariable functions. That is our goal for this section.

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Definition

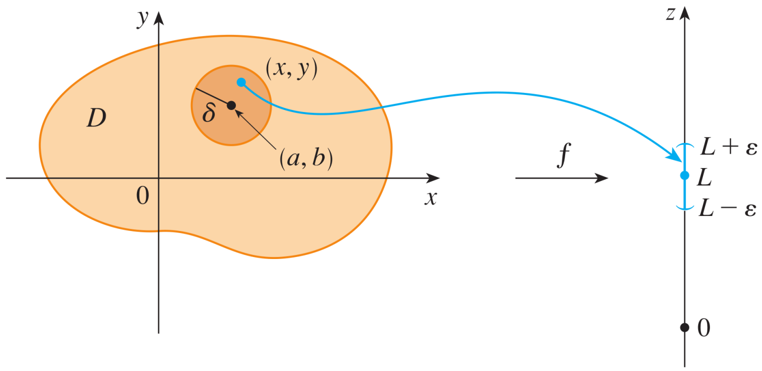
Let f be a function of two variables whose domain contains points arbitrarily close to (a, b). We say that the **limit of** $f(x, y)$ **as** (x, y) **approaches** (a, b) is L if $f(x, y)$ is arbitrarily close to L whenever (x, y) is close enough to (a, b). We write this as follows:

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

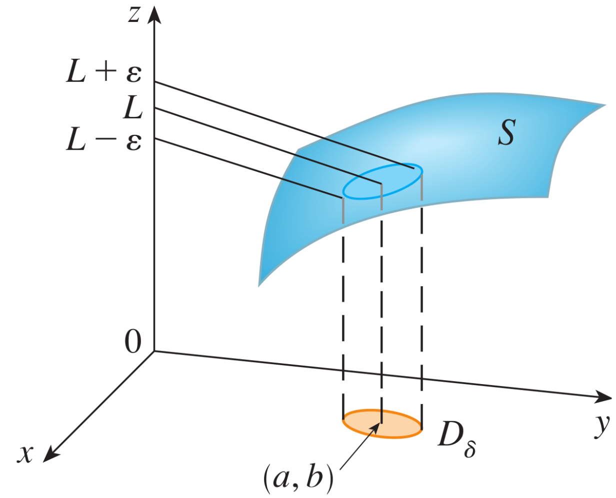
This definition is analogous to the definition for single-variable functions.

The Picture

We can visualize this definition with the following images:



The Picture, cont.



The Picture, cont.

Whenever (x, y) is in the small circle (of radius δ) around (a, b), we have that $f(x, y)$ is close to L, in this case between $L - \varepsilon$ and $L + \varepsilon$.

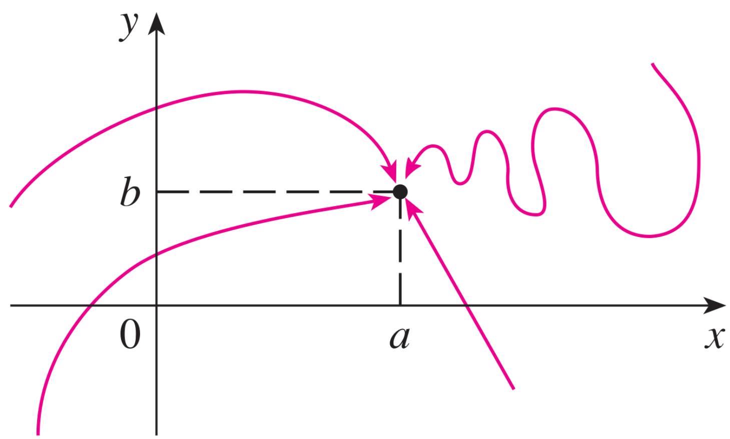
One-Sided Limits?

These pictures give us an idea of what it means to have a limit, but they tell us hardly anything about how to calculate one, or even to determine if one exists! Our next goal is to work out how to do just that.

Recall that with single-variable functions we sometimes determined, say, $\lim_{x \to a} f(x)$ by first evaluating each of its one-sided limits and comparing them. Put another way, we evaluated the limit of f along all possible continuous paths x could take to a. If they were the same, the limit existed, otherwise it did not. Maybe we could try something similar here?

A Problem

When evaluating $\lim_{x \to a} f(x)$, there are only two ways for x to approach a: from the left or from the right. But for a multivariable function, there are infinitely-many ways for (x, y) to approach (a, b):



A Problem?

For the limit to exist, the limits along every possible path toward (a, b) must agree. This may appear daunting at first; one certainly cannot check infinitely-many paths for agreement. However, there's a handy flipside to this: if any two paths toward (a, b) have *different* limits, then the overall limit doesn't exist!

Example

Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

First, let (x, y) approach $(0, 0)$ along the path $x = 0$. We have:

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{-y^2}{y^2} = -1$$

On the other hand, let (x, y) approach $(0, 0)$ along the path $y = 0$:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2}{x^2} = 1$$

Since these limits do not agree, the overall limit does not exist.

Paths to Try

This worked out rather neatly, but how did I know to try the paths $x = 0$ and $y = 0$? First, when approaching $(0, 0)$, there are several paths that we typically try:

$$x = 0$$
$$y = 0$$
$$x = y$$
$$x = -y$$
$$x = y^2$$
$$y = x^2$$

Second, try to use paths toward (a,b) that create nice cancellations when plugging in. For example, the paths $x = 0$ and $y = 0$ toward $(0, 0)$ in the previous problem made terms cancel nicely after plugging in.

Example

Does $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exist?

First, note that the path $x = 0$ runs through $(0, 0)$, and plugging in $x = 0$ gives a very nice limit:

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{xy^2}{x^2+y^4} = \lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{0}{y^4} = 0$$

Second, the path $x = y^2$ also runs through $(0, 0)$, and plugging this into the limit gives a very nice cancellation:

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} \frac{xy^2}{x^2+y^4} = \lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} \frac{y^4}{y^4+y^4} = \frac{1}{2}$$

Therefore, the limit $\boxed{\text{does not exist}}$ (It's worth noting that the path $y = \sqrt{x}$ would also be a good one to try).

Showing a Limit Does Exist

We are getting quite adept at showing a limit does *not* exist, but how do we show that one does?

One method involves using the formal definition of a limit, which we won't cover here because it can be quite invloved. Sometimes this is absolutely necessary, but it's generally better to use more sophisticated methods when possible.

Another suggestion is to use methods you learned in single-variable calculus: the squeeze theorem, conjugation, and direct evaluation of continuous functions. The latter requires some more explanation.

Continuity

A function f of two variables is called **continuous** at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

i.e. the limit of the function is the the actual value of the function at (a, b). We say that f is continuous (on its domain) if it is continuous at every (a, b) in its domain.

Other than piecewise-defined functions, every function you encounter in this course will be continous on its domain. Thus, if we wish to evaluate $\lim_{(x,y)\to(a,b)} f(x, y)$ and (a, b) is in the domain of f, we can evaluate the limit simply by plugging in!

Example

Evaulate $\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y)$ or show that it does not exist.

Note that $(1, 2)$ is in the domain of $x^2y^3 - x^3y^2 + 3x + 2y$, a function defined and continuous everywhere (since it is a polynomial). Thus, we have:

$$\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y) = 2^3 - 2^2 + 3 + 4 = \boxed{11}$$

Calculating Limits: a Flowchart

To determine whether $\lim_{(x,y)\to(a,b)} f(x,y)$ exists (if f is not a piecewise-defined function):

1. Can you plug (a,b) into f? If so, (and f is not a piecewise-defined function) the limit exists, and is equal to $f(a,b)$. If not, continue:
2. Let $(x,y) \to (a,b)$ along a few paths through (a,b), chosen as thoughtfully as you can. Do you get different limits along different paths? If so, the limit does not exist. If you seem to be getting the same thing every time, continue:
3. Try to show the limit exists by using the squeeze theorem, conjugation, or any other method for testing limits that you learned in single-variable calculus.

One Final Note

Limits can be defined similarly for functions of more than two variables. For example, $\lim_{(x,y,z)\to(a,b,c)} f(x,y,z)$ is L if $f(x,y,z)$ is close to L whenever (x,y,z) is close to (a,b,c). This limit exists if and only if the limits of f along every path in $\mathbb{R}^3$ to (a,b,c) agree.

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Exercises

1. Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ or show that it does not exist.
2. Evaluate $\lim_{(x,y)\to(2,-1)} \frac{x^2y+xy^2}{x^2-y^2}$ or show that it does not exist.
3. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$ or show that it does not exist [Hint: it exists; try some of the other methods for showing it exists mentioned in the slides.].

Solutions

1. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist; try the paths $x = 0$ and $x = y$ to see.
2. $\lim_{(x,y)\to(2,-1)} \frac{x^2y+xy^2}{x^2-y^2} = \frac{-2}{3}$; plug in to see this.
3. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = 2$; use conjugation to see this.