14.1: Functions of Several Variables

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Chapter Overview

With this chapter, we have completed the first act of the course (the calculus of curves in $\mathbb{R}^3$) and arrived at the second: tackling the calculus of surfaces in $\mathbb{R}^3$. In this chapter we will study differentiation; the next will cover integration.

In order to study this calculus, we first need to work out what sorts of functions have surfaces as their graphs. We will see that the appropriate functions are the namesake of the course: multivariable functions. We will begin by defining what multivariable functions are and how to graph them. We will then essentially cover all the topics of single-variable calculus for these new functions: we will calculate limits, derivatives, and equations of tangent planes (extensions of tangent lines), before tackling optimization.

Section Overview

In this first section, we begin by introducing multivariable functions. We will define them, learn to find their domains and evaluate them at various points, and cover the various ways they can be represented (algebraically, with a table, with a graph, with level curves). In particular, we will see that the graphs of multivariable functions are surfaces, thus demonstrating that studying the calculus of surfaces amounts to studying the calculus of these functions.

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Definition

A **multivariable function of** n **variables** $f(x_1, x_2, \ldots, x_n)$ is a rule that assigns a single real number, or output, to each tuple $(x_1, \ldots, x_n)$ of real numbers. Its **domain** is the maximal set of inputs that f can accept.

The x_i are called **independent variables**. In the case of two variables, we will often write $z = f(x, y)$, in which case z is the **dependent variable** (x and y are the independent variables).

Examples

Such functions arise naturally in many applications. For example, the wind-chill temperature W outside in the winter depends on both the actual air temperature T and wind speed v. We can express this relationship as the function $W(T, v)$.

Or, the rendering time R of a frame in an animated movie depends on the processor speed C of the rendering computer, its graphics card speed G, the amount r of RAM it has, its operating system S, and the size P of its power supply (and probably many other factors). We can express this relationship as $R(C, G, r, S, P)$.

Tabular Presentation

Multivariable functions can be presented in many different ways. For example, several values of a two-variable function can be given in a table. The function $W(T, v)$ from above could be represented in the following table, whose entries are the wind-chill temperature for a given air temperature and wind speed:

Wind speed (km/h)

Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

Tabular Presentation, cont.

So, for example, the wind-chill temperature when the air temperature is -15° C and the wind speed is 40 km/h can be read off as:

$$W(-15, 40) = -27^\circ\text{C}$$

Or, the wind-chill temperature when the air temperature is -30° C and the wind speed is 10 km/h is:

$$W(-30, 10) = -39^\circ\text{C}$$

Algebraic Presentation

Most often we will present our functions alegbraically. For example, $f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$ and $g(x,y,z) = \frac{\ln(x-y+z)}{x-2}$ are multivariable functions of two and three variables, respectively.

We can evaluate, say $f(3,2)$ and $g(-1,1,3)$ analogously to the way we would with single-variable functions:

$$f(3,2) = \frac{\sqrt{3+2+1}}{3-1} = \frac{\sqrt{6}}{2}$$

and

$$g(-1,1,3) = \frac{\ln(-1-1+3)}{-1-2} = \frac{\ln(1)}{-3} = 0$$

Important Aside: Domain

Of course, not every ordered tuple can be plugged into a given multivariable function. For example, given

$$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

as above, note that

$$f(0, -3) = \frac{\sqrt{0 - 3 + 1}}{0 - 1}$$

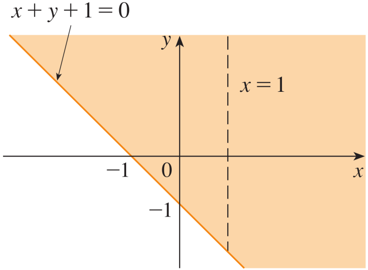
is undefined. An important question to answer in general is which ordered tuples *can* be plugged into a given multivariable function, i.e., what the function's domain is. Moreover, what is an effective way to communicate this domain to others?

In the case of a function of two variables, the domain is easiest to specify with a graph.

Example

Find and sketch the domains D_1 and D_2 of $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$ and $g(x, y) = x \ln(y^2 - x)$, respectively.

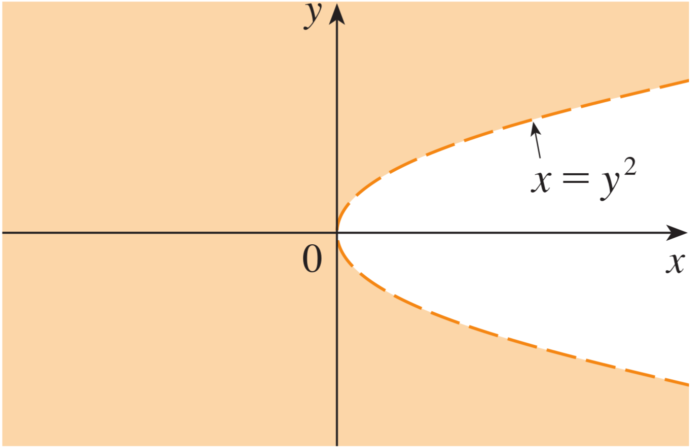
Using the usual methods of working out the domain of a single-variable function, we see that for f to be defined we must have both $x - 1 \neq 0$ and $x + y + 1 \geq 0$. That is, $x \neq 1$ and $y \geq -x - 1$. The graph of the set satisfying both of these conditions (i.e. the graph of D_1) is the following:



D_1

Example, cont.

Similarly, for g to be defined we must have $y^2 - x > 0$, i.e. $x < y^2$. The graph of this set looks like this:



D_2

Graphs

The final way we will present a *two-variable* function is as a graph. Given a two-variable function $f(x, y)$, its **graph** is the set of all points (x, y, z) such that $z = f(x, y)$, for (x, y) in the domain of f.

In particular, the graph of a two-variable function is a *surface* in $\mathbb{R}^3$, as we shall see.

Example

Sketch the graph of $f(x, y) = \sqrt{9 - x^2 - y^2}$.

First, note that the domain of f is all tuples (x, y) such that $9 - x^2 - y^2 \geq 0$, i.e. such that $x^2 + y^2 \leq 9$. These are all the points in the xy-plane inside and on a circle of radius 3 centered at the origin.

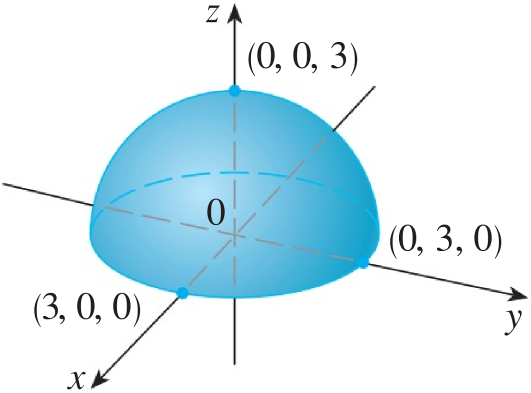
Next, note that the graph of f has equation $z = \sqrt{9 - x^2 - y^2}$. So, all the values of z here must be either positive or zero.

Finally, if we square both sides of the latter equation and rearrange terms, we have:

$$x^2 + y^2 + z^2 = 9$$

Example

Thus, combining all of this information, the graph of f is the top half of a sphere of radius 3 centered at the origin:



$$f(x, y) = \sqrt{9 - x^2 - y^2}$$

Complications

In the previous example, we got lucky. The graph of f fell in our laps after rearranging a bit because it was a surface we encountered previously. But what if we had a more complicated function, such as $f(x, y) = \sin(xy)$? How would we sketch the graph of f? One answer lies in the concept of **level curves**.

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Definition

The **level curves** of a function f of *two* variables are the curves $f(x, y) = k$ for k a constant.

A level curve of the graph of f is the same as a trace of the graph parallel to to the xy-plane. That is, a trace in $z = k$. You can think of such curves as analogous to the elevation lines on a topographical map.

A collection of level curves charted simultaneously on the same set of axes is called a **contour map**.

Example

In $\mathbb{R}^2$, sketch the level curves of $f(x, y) = \sqrt{9 - x^2 - y^2}$ for $k = 0, 1, 2, 3$.

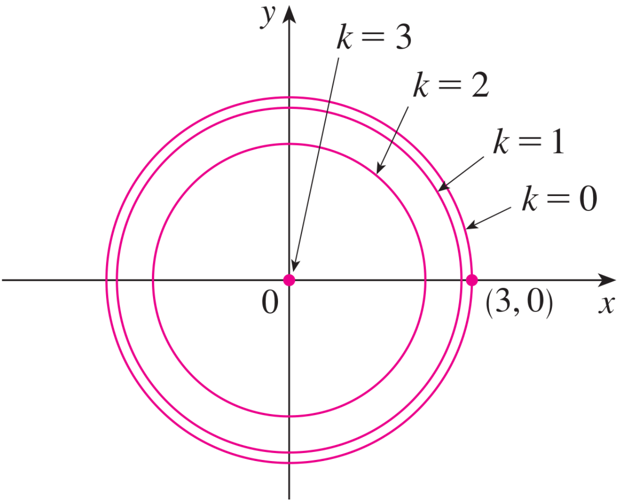
We saw above that the graph of f is the top half of a sphere of radius 3 centered at the origin. The level curves of f are simply horizontal cross-sections of its graph, which we know to be circles. Let's verify this:

The level curves for a given k are $k = \sqrt{9 - x^2 - y^2}$, or:

$$x^2 + y^2 = 9 - k^2$$

i.e. a circle of radius $\sqrt{9 - k^2}$ centered at the origin. Here's what they look like in each of the cases above:

Example, cont.



Four level curves of $f(x, y) = \sqrt{9 - x^2 - y^2}$

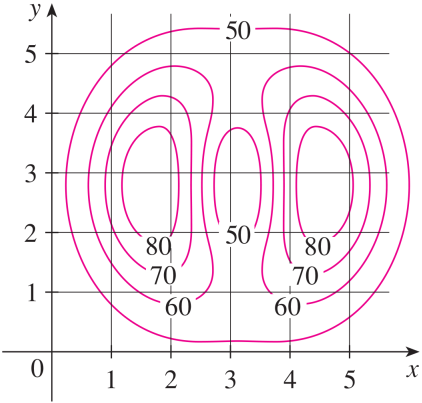
Graphs and Estimation

Just as we did with traces in chapter 12, we can place level curves in $\mathbb{R}^3$ and stitch them together to sketch the graph of a two-variable function.

We can also use contour maps on their own to give us information about a function. For example, we can use them to estimate values of a function, as in the next example.

Example

The following is a contour map of a continuous function f:



Use this to estimate $f(1, 3)$ and $f(4, 5)$.

Example, cont.

We note that $(1, 3)$ is between the level curves with z-values 70 and 80. So, we estimate $\boxed{f(1, 3) \approx 72}$. Similarly, we estimate $\boxed{f(4, 5) \approx 57}$.

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Exercises

1. Using the wind-chill chart in the slides above, calculate the wind-chill temperature when the air temperature is -20° C and the wind speed is 25 km/h.
2. Find and sketch the domain of $f(x, y) = \sqrt{9 - x^2 - y^2}$.
3. Describe the graph of $g(x, y) = 6 - 3x - 2y$.
4. Sketch the level curves of $g(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

Solutions

1. $W(-20, 25) = -32^{\circ}$C
2. The domain of f is the interior and boundary of the circle of radius 3 centered at the origin, in $\mathbb{R}^2$.
3. The graph of g is a plane through (for example) the points $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 6)$ with normal vector $\langle 3, 2, 1 \rangle$.
4. The indicated level curves of g are all are lines of slope $\frac{-3}{2}$ with y-intercepts $(0, 6)$, $(0, 3)$, $(0, 0)$, and $(0, -3)$, respectively.