13.4: Motion in Space: Velocity and Acceleration

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Fall 2021

Overview

Before we finish out this chapter, we examine one more application of differentiation and integration of vector functions. As we have seen, the graph of a vector function $\vec{r}(t)$ is a space curve C. We could think of this curve as the path along which a particle or an object travels. A natural question to ask, then, is what the derivative of $\vec{r}(t)$ physically represents. The answer is exactly what you would expect, based on your knowledge of single-variable calculus.

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Velocity and Acceleration

Suppose the vector function $\vec{r}(t)$ describes an object's position in space at time t. The object's velocity is then given by:

$$\vec{v}(t) = \vec{r}\,'(t)$$

and its acceleration is given by

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

Both are completely analogous to single-variable calculus, but take care to note that acceleration and velocity are both vectors, which makes sense: they have both a magnitude and direction. For example, velocity measures both how fast the object is traveling, i.e. its speed (magnitude of the velocity vector) and what direction it is traveling in (direction of the velocity vector).

Example

Find the velocity $\vec{v}(t)$, acceleration $\vec{a}(t)$, and speed $\sigma(t)$ at $t = 2$ of a particle with position vector $\vec{r}(t) = \left\langle t^2, e^t, te^t \right\rangle$.
From the definitions, we have:

$$\vec{v}(t) = \vec{r}\,'(t) = \left\langle 2t, e^t, e^t + te^t \right\rangle$$
$$\vec{a}(t) = \vec{v}\,'(t) = \left\langle 2, e^t, 2e^t + te^t \right\rangle$$

Thus, the velocity of the particle at $t = 2$ is:

$$\boxed{\vec{v}(2) = \left\langle 4, e^2, 3e^2 \right\rangle}$$

the acceleration of the particle at $t = 2$ is:

$$\boxed{\vec{a}(2) = \left\langle 2, e^2, 4e^2 \right\rangle}$$

and the speed of the particle at $t = 2$ is:

$$\boxed{\sigma(2) = \left|\vec{v}(2)\right| = \sqrt{16 + 10e^4} \approx 23.706}$$

Example

Suppose a particle's acceleration is given by $\vec{a}(t) = \left\langle 4t, 6t, e^t \right\rangle$. If its initial velocity is $\vec{v}(0) = \langle 1, -1, 1 \rangle$, find its velocity $\vec{v}(t)$ at time t.

Since $\vec{a}(t) = \vec{v}'(t)$, we have:

$$\begin{aligned} \vec{v}(t) &= \int \vec{a}(t)\, \mathrm{d}t = \int \left\langle 4t, 6t, e^t \right\rangle \mathrm{d}t \\ &= \left\langle 2t^2, 3t^2, e^t \right\rangle + \vec{C} \end{aligned}$$

for some constant vector $\vec{C}$. Now we must find $\vec{C}$.

Example, cont.

On the one hand, we are told that $\vec{v}(0) = \langle 1, -1, 1 \rangle$. On the other hand, plugging 0 into $\vec{v}(t)$ above gives:

$$\vec{v}(0) = \langle 0, 0, 1 \rangle + \vec{C}$$

Therefore, equating these expressions for $\vec{v}(0)$ gives:

$$\begin{aligned} \langle 1, -1, 1 \rangle &= \langle 0, 0, 1 \rangle + \vec{C} \\ \Rightarrow \qquad \vec{C} &= \langle 1, -1, 0 \rangle \end{aligned}$$

Thus:

$$\boxed{\vec{v}(t) = \left\langle 2t^2, 3t^2, e^t \right\rangle + \langle 1, -1, 0 \rangle = \left\langle 2t^2 + 1, 3t^2 - 1, e^t \right\rangle}$$

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Exercises

1. Suppose that the initial position of the particle in the previous example is $\vec{r}(0) = \langle 1, 0, 0 \rangle$. Find its position $\vec{r}(t)$ at time t.
2. Calculate the velocity $\vec{v}(t)$, acceleration $\vec{a}(t)$, and speed $\sigma(t)$ at $t = 0$ of a particle with position $\vec{r}(t) = \left\langle t, 2\cos(t), \sin(t) \right\rangle$.
3. Calculate the velocity $\vec{v}(t)$ and position $\vec{r}(t)$ of a particle with acceleration $\vec{a}(t) = \left\langle \sin(t), 2\cos(t), 6t \right\rangle$, initial velocity $\vec{v}(\pi) = \left\langle 2, 1, 3\pi^2 \right\rangle$, and initial position $\vec{r}\left(\frac{\pi}{2}\right) = \left\langle -1, \frac{\pi}{2}, \frac{\pi^3}{8} \right\rangle$.

Solutions

1. $\vec{r}(t) = \left\langle \frac{2}{3}t^3 + t + 1, t^3 - t, e^t - 1 \right\rangle$
2. $\vec{v}(0) = \langle 1, 0, 1 \rangle$; $\sigma(0) = \left| \vec{v}(0) \right| = \sqrt{2}$; and $\vec{a}(0) = \langle 0, -2, 0 \rangle$
3. $\vec{v}(t) = \left\langle -\cos(t) + 1, 2\sin(t) + 1, 3t^2 \right\rangle$ and
   $\vec{r}(t) = \left\langle -\sin(t) + t - \frac{\pi}{2}, -2\cos(t) + t, t^3 \right\rangle$