# 13.3: Arc Length

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## Overview

Now that we have learned how to calculate derivatives and integrals of vector functions, we examine applications of these. We begin with measuring the length of an arc on a space curve, extending the idea from single-variable calculus.

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## A Refresher and a Connection

Recall that if we are given a curve in the $xy$-plane with parametric equations $x = f(t)$ and $y = g(t)$ with $f$ and $g$ differentiable, the length $L$ of the arc between $t = a$ and $t = b$ is given by the formula:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

To put it another way, if $\vec{r}(t) = \langle f(t), g(t) \rangle$ is a vector function in $\mathbb{R}^2$ with graph $C$, the length of the arc on $C$ between $t = a$ and $t = b$ is given as above.

## Generalizing

How might we generalize the formula on the previous slide to $\mathbb{R}^n$ for $n \geq 3$? Let's adjust it slightly for the answer. If $\vec{r}(t) = \langle f(t), g(t) \rangle$, then $\vec{r}'(t) = \langle f'(t), g'(t) \rangle$. The length $L$ of the arc on the graph of $\vec{r}(t)$ between $t = a$ and $t = b$ is then:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$
$$= \int_a^b \left|\vec{r}'(t)\right| dt$$

It turns out that the latter formula holds in general. Thus, once and for all: if $\vec{r}(t)$ is a vector-valued function in $\mathbb{R}^n$ for *any* $n$, the length of the arc on the graph of $\vec{r}(t)$ between $t = a$ and $t = b$, where $a \leq b$, is:

$$\boxed{L = \int_a^b \left|\vec{r}'(t)\right| dt}$$

## Example

Find the length $L$ of the arc lying on the graph of $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ between $(1, 0, 0)$ and $(1, 0, 2\pi)$.

From above, we have:

$$L = \int_a^b \left|\vec{r}'(t)\right| dt$$

To use this formula, we must determine $a$ and $b$. Well, note that the tip of $\vec{r}(t)$ is at $(1, 0, 0)$ when $t = 0$ by comparing the third component of $\vec{r}(t)$ to the third coordinate of $(1, 0, 0)$; and the tip of $\vec{r}(t)$ is at $(1, 0, 2\pi)$ when $t = 2\pi$ similarly. Thus, we have:

$$L = \int_0^{2\pi} \sqrt{(-\sin(t))^2 + \cos^2(t) + 1}\, dt$$
$$= \int_0^{2\pi} \sqrt{2}\, dt$$
$$= \boxed{2\pi\sqrt{2}}$$

## Toward The Arc Length Function

At this point, we can quickly find the length of an arc between two points on the graph of a vector function. Here's an interesting related question, which will motivate our work through the rest of the section:

*Find the point $P$ on the graph of the vector function*

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

*which is four units away from the point $(1, 0, 0)$ in the direction of increasing $t$.*

So, rather than find the distance between two given points, start at a given point, go a prescribed distance away, and find the second endpoint of the arc. We will solve this problem in two ways, one of which introduces a new concept, important in its own right: the **arc length function**.

## First Solution

*Find the point $P$ on the graph of the vector function*

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

*which is four units away from the point $(1, 0, 0)$ in the direction of increasing $t$.*

Note first from the previous example that $(1, 0, 0)$ is the point on the graph of $\vec{r}(t)$ corresponding to $t = 0$. Beginning at this point, we sweep out an arc on the graph of $\vec{r}(t)$ of length four, and finish at some $t$-value, say $t = b$. Therefore, using the arc length formula we can set up the following equations:

$$4 = \int_0^b \sqrt{2}\, dt$$
$$= t\sqrt{2}\Big|_0^b = b\sqrt{2}$$

## First Solution, cont.

Thus, solving for $b$, we have

$$b = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

and hence $P$ is at the tip of $\vec{r}(2\sqrt{2}) = \left\langle \cos(2\sqrt{2}), \sin(2\sqrt{2}), 2\sqrt{2} \right\rangle$:

$$\boxed{P = \left(\cos(2\sqrt{2}), \sin(2\sqrt{2}), 2\sqrt{2}\right)}$$

## Follow-Up

Great! Our motivating problem is really just a twist on the first type of problem we learned to solve.

Now let's take a slightly different perspective, to introduce a new concept: the **arc length function**.

## Second Solution

*Find the point $P$ on the graph of the vector function*

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

*which is four units away from the point $(1, 0, 0)$ in the direction of increasing $t$.*

Again, the point $(1, 0, 0)$ on the graph of $\vec{r}(t)$ corresponds to $t = 0$. Now we proceed slightly differently: note that, using the arc length formula, we have that the length of any arc on the graph of $\vec{r}(t)$ starting at $t = 0$ and ending at some second, arbitrary $t$-value is given by the function:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{2}\, du = u\sqrt{2}\Big|_0^t \\ &= t\sqrt{2} \end{aligned}$$

This is called the **arc length function of $\vec{r}(t)$ starting at $t = 0$**.

## Second Solution, cont.

The arc length function truly is a function: given any second $t$-value $t = b$ with $b \geq a$, we can evaluate $s(b)$ and obtain the length of the arc on the graph of $\vec{r}(t)$ between $t = 0$ and $t = b$.

## Second Solution, cont.

How does this help us toward an answer? Here's the key insight:

From the equation $s(t) = t\sqrt{2}$, we can solve for $t$ in terms of the variable $s$:

$$t = \frac{s}{\sqrt{2}}$$

Now we can swap out $t$ in $\vec{r}(t)$ to obtain a new **parametrization** of the graph of $\vec{r}(t)$:

$$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$$

This is called the **reparametrization of $\vec{r}$ with respect to arc length**.

## Second Solution, cont.

This parametrization of the graph of $\vec{r}$ is interesting in its own right. Rather than finding the points on the graph of $\vec{r}$ by relying on an outside parameter $t$, one can find them using only their distance $s$ from $(1, 0, 0)$!

So, for example, to find what point is four units away from $(1, 0, 0)$ on the graph of $\vec{r}(s)$ in the direction of increasing $t$, we plug in $s = 4$:

$$\vec{r}(4) = \left\langle \cos\left(\frac{4}{\sqrt{2}}\right), \sin\left(\frac{4}{\sqrt{2}}\right), \frac{4}{\sqrt{2}} \right\rangle$$

and obtain the endpoint

$$\boxed{\left(\cos(2\sqrt{2}), \sin(2\sqrt{2}), 2\sqrt{2}\right)}$$

as desired.

## The Arc Length Function

Generally, given a vector function $\vec{r}(t)$, the length of the arc on the graph of $\vec{r}(t)$ between $t = a$ and any second value of $t$ is given by the function:

$$\boxed{s(t) = \int_a^t \left|\vec{r}\,'(u)\right| \mathrm{d}u}$$

This is called the **arc length function of $\vec{r}(t)$ starting at** $t = a$. Notice: $u$ is a *dummy variable* of integration, and $t$ is the actual variable of interest to the function $s(t)$.

## Dummy Variables

What is a *dummy variable*? It's an extra variable that we introduce whose sole purpose is to help us evaluate the integral. The reason it's there is because when we evaluate, say, $s(b)$, we only want to plug $b$ into the upper bound on the integral, not the function under the integral sign. Indeed, the arc length formula tells us that the length $L$ of the arc on the graph of $\vec{r}(t)$ between $t = a$ and $t = b$ is

$$L = \int_a^b \left|\vec{r}\,'(t)\right| \mathrm{d}t = \int_a^b \left|\vec{r}\,'(u)\right| \mathrm{d}u$$

and NOT

$$\require{cancel}\cancel{L = \int_a^b \left|\vec{r}\,'(b)\right| \mathrm{d}b}$$

Introducing a dummy variable assures that $b$ will *only* be plugged into $s(t)$ where it's supposed to: in the upper bound of the integral.

## Summary of the Second Method

Let's put this all together, one last time:

Given a vector function $\vec{r}(t)$, its graph $C$, and an initial point $(u, v, w)$ on $C$, to find the point on $C$ that is $\ell$ units away from $(u, v, w)$ as $t$ increases:

1. Find the value of $t$, say $a$, such that $\vec{r}(a) = \langle u, v, w \rangle$.
2. Calculate the arc length function $s(t)$ of $\vec{r}(t)$ starting at $t = a$.
3. Solve the arc length function for $t$, to write $t = f(s)$.
4. Plug $t = f(s)$ into $\vec{r}(t)$ to obtain $\vec{r}(s)$.
5. Plug $\ell$ in for $s$ in $\vec{r}(s)$ to get your answer.

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## Exercises

1. Find the length $L_1$ of the curve given by the graph of $\vec{r}(t) = \left\langle 2t, t^2, \frac{t^3}{3} \right\rangle$ with $0 \leq t \leq 1$.
2. Find the arc length function $s(t)$ for the curve given by the graph of $\vec{r}(t) = \left\langle e^t \sin(t), e^t \cos(t), \sqrt{2}e^t \right\rangle$ measured from $(0, 1, \sqrt{2})$ in the direction increasing $t$.
3. Reparametrize the curve in the previous problem with respect to arc length.
4. Find the point $P$ four units from $(0, 1, \sqrt{2})$ in the direction of increasing $t$, along the curve from the previous two problems.

## Solutions

1. $L_1 = \dfrac{7}{3}$
2. $s(t) = 2e^t - 2$
3. $\vec{r}(s) = \left\langle e^{\ln\left(\frac{s+2}{2}\right)} \sin\left(\ln\left(\frac{s+2}{2}\right)\right), e^{\ln\left(\frac{s+2}{2}\right)} \cos\left(\ln\left(\frac{s+2}{2}\right)\right), \sqrt{2}e^{\ln\left(\frac{s+2}{2}\right)} \right\rangle$
4. $P = \left(e^{\ln(3)} \sin\left(\ln(3)\right), e^{\ln(3)} \cos\left(\ln(3)\right), \sqrt{2}e^{\ln(3)}\right)$