# 13.2: Derivatives and Integrals of Vector Functions

Julia Jackson

Department of Mathematics
The University of Oklahoma

Fall 2021

## Overview

In the previous section we defined the concept of a vector function (i.e. a function that takes a real number and returns a vector), and learned that the graphs of such functions in $\mathbb{R}^3$ are space curves. Therefore, the calculus of space curves in $\mathbb{R}^3$ amounts to the calculus of these vector functions.

We began moving toward this calculus by defining the limit of a vector function. We now continue in that vein, and cover the fundamentals of the calculus of space curves: derivatives and integrals.

## Table of Contents



## The Derivative

Let $\vec{r}(t)$ be a vector-valued function. Its **derivative** is defined as follows:

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Note that this is analogous to the definition of the derivative of a single-variable real-valued function $f(x)$. In fact, one can show that directly from this definition that if $\vec{r}(t) = \left\langle f(t), g(t), h(t) \right\rangle$ with $f$, $g$, and $h$ differentiable, then

$$\vec{r}\,'(t) = \left\langle f'(t), g'(t), h'(t) \right\rangle$$

# Example

Calculate the derivative of $\vec{r}(t) = \left\langle 1+t^3, te^{-t}, \sin(2t) \right\rangle$ at $t = 0$.

From the previous slide, we have:

$$\begin{aligned}\vec{r}\,'(t) &= \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(1+t^3), \frac{\mathrm{d}}{\mathrm{d}t}(te^{-t}), \frac{\mathrm{d}}{\mathrm{d}t}\sin(2t) \right\rangle \\ &= \left\langle 3t^2, e^{-t} - te^{-t}, 2\cos(2t) \right\rangle\end{aligned}$$

Thus, plugging in $t = 0$, we have:

$$\boxed{\vec{r}\,'(0) = \langle 0, 1, 2 \rangle}$$

# Graphical Interpretation

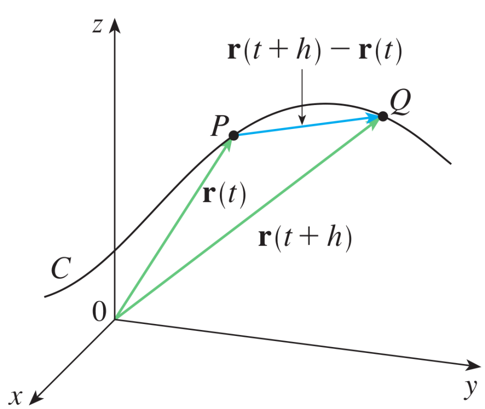
Given a real-valued function $f(x)$, we know that the derivative of $f$ at $x = a$ represents the slope of the line tangent to $f$ at the point $(a, f(a))$. What can we say about the derivative of a vector-valued function $\vec{r}(t)$?

# Graphical Interpretation, cont.

Recall that for a vector function $\vec{r}(t)$, we have:

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Consider the following sketch of $C$, the graph of $\vec{r}(t)$:




# Graphical Interpretation, cont.

As $h$ approaches 0, the point $Q$ in the previous slide approaches the point $P$, and thus the secant vector $\vec{r}(t+h) - \vec{r}(t)$ approaches a vector tangent to $C$ at $P$. Multiplying by $\frac{1}{h}$ only scales the secant vector in this process, and thus:

$$\boxed{\vec{r}\,'(t) \text{ represents a vector tangent to the graph of } \vec{r}(t) \text{ at a given } t}$$

## Example

Find a vector $\vec{w}$ that lies tangent to the graph of the vector function $\vec{r}(t) = \langle 1 + t^3, te^{-t}, \sin(2t) \rangle$ at the point $(1, 0, 0)$.

From the previous slide, we know that one such vector is the derivative of $\vec{r}(t)$ at the point $(1, 0, 0)$. So, the first thing we need to know is when the graph of $\vec{r}(t)$ passes through the point $(1, 0, 0)$, i.e., what $t$-value(s) give $\vec{r}(t) = \langle 1 + t^3, te^{-t}, \sin(2t) \rangle = \langle 1, 0, 0 \rangle$. Well, from the first components of each we see that $1 + t^3 = 1$ gives $t = 0$. A quick calculation verifies that $\vec{r}(0) = \langle 1, 0, 0 \rangle$.

Therefore, a vector tangent to the graph of $\vec{r}(t)$ at the point $(1, 0, 0)$ is:

$$\vec{w} = \vec{r}'(0) = \left\langle 3t^2, e^{-t} - te^{-t}, 2\cos(2t) \right\rangle \Big|_{t=0} = \boxed{\langle 0, 1, 2 \rangle}$$

## The Unit Tangent Vector

For calculations, we will sometimes want the tangent vector to have unit length (i.e. length 1), so we construct the **unit tangent vector**:

$$\vec{T}(t) := \frac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|}$$

First, notice that this vector is also tangent to the graph of $\vec{r}(t)$, as it is a scalar multiple of $\vec{r}'(t)$. Furthermore, it has length 1, as:

$$\left|\vec{T}(t)\right| = \left|\frac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|}\right| = \frac{\left|\vec{r}'(t)\right|}{\left|\vec{r}'(t)\right|} = 1$$

## Example

Find a unit vector $\vec{u}$ that lies tangent to graph of $\vec{r}(t) = \langle 1 + t^3, te^{-t}, \sin(2t) \rangle$ at the point $(1, 0, 0)$.

In the previous example, we saw that a vector tangent to $\vec{r}(t)$ at the point $(1, 0, 0)$ is $\vec{r}'(0) = \langle 0, 1, 2 \rangle$. So, a *unit* tangent vector is:

$$\vec{u} = \frac{\vec{r}'(0)}{\left|\vec{r}'(0)\right|} = \frac{\langle 0, 1, 2 \rangle}{\left|\langle 0, 1, 2 \rangle\right|} = \frac{\langle 0, 1, 2 \rangle}{\sqrt{0^2 + 1^2 + 2^2}} = \boxed{\left\langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle}$$

## Connection to Slope and the Tangent Line

In single-variable calculus, the derivative $f'(a)$ of a function $f(x)$ at $x = a$ gives the slope of the line tangent to the graph of $f(x)$ at the point $(a, f(a))$. Using this information, we can then give an equation for this tangent line.

Here we have a similar situation: the derivative $\vec{r}'(a)$ of $\vec{r}(t)$ at $t = a$ gives a vector tangent to the graph of $\vec{r}(t)$ at the point corresponding to $\vec{r}(a)$. This tangent vector can be thought of as a direction vector of the line tangent to the graph of $\vec{r}(t)$ at $t = a$.

## Example

Give a set of parametric equations for the line $L$ that lies tangent to the graph of $\vec{r}(t) = \langle 1 + t^3, te^{-t}, \sin(2t) \rangle$ at the point $(1, 0, 0)$.

We know from the first example above that a vector tangent to the graph of $\vec{r}(t)$ at the point $(1, 0, 0)$ is $\langle 0, 1, 2 \rangle$. Therefore, a direction vector for $L$ is $\langle 0, 1, 2 \rangle$. Furthermore, a point on $L$ is $(1, 0, 0)$. Thus, by our work in section 12.5, a set of parametric equations for $L$ is:

$$\boxed{\begin{aligned} x(t) &= 1 \\ y(t) &= t \\ z(t) &= 2t \end{aligned}}$$

## Differentiation Rules

Just as with real-valued functions, there are several differentiation rules that may come in handy. Let $\vec{u}(t)$ and $\vec{v}(t)$ be differentiable vector functions, let $c$ be a scalar, and let $f(t)$ be a real-valued function. We have:

1. $\dfrac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \vec{u}'(t) + \vec{v}'(t)$
2. $\dfrac{d}{dt}\left(c\vec{u}(t)\right) = c\vec{u}'(t)$
3. $\dfrac{d}{dt}\left(f(t)\vec{u}(t)\right) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
4. $\dfrac{d}{dt}\left(\vec{u}(t) \cdot \vec{v}(t)\right) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
5. $\dfrac{d}{dt}\left(\vec{u}(t) \times \vec{v}(t)\right) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
6. $\dfrac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\vec{u}'(f(t))$

## Table of Contents



## Definition

The **definite integral** of a continuous vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is defined analogously to the definite integral of a continuous real function, as follows:

$$\begin{aligned} \int_a^b \vec{r}(t)\, dt &= \lim_{n\to\infty} \sum_{i=1}^{n} \vec{r}(t_i)\Delta t \\ &= \lim_{n\to\infty} \left\langle \sum_{i=1}^{n} f(t_i)\Delta t, \sum_{i=1}^{n} g(t_i)\Delta t, \sum_{i=1}^{n} h(t_i)\Delta t \right\rangle \\ &= \left\langle \int_a^b f(t)\, dt, \int_a^b g(t)\, dt, \int_a^b h(t)\, dt \right\rangle \end{aligned}$$

Unlike the derivative, there is no nice graphical interpretation of the definite integral of a vector function.

## The Fundamental Theorem

Let $\vec{R}(t)$ be an antiderivative for $\vec{r}(t)$. Then, by the previous slide, the Fundamental Theorem of Calculus carries over for vector functions, and says:

$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

## Example

Let $\vec{r}(t) = \langle 2\cos(t), \sin(t), 2t\rangle$. Evaluate $\int_0^{\pi/2} \vec{r}(t)\,\mathrm{d}t$.

By the fundamental theorem, we have:

$$\begin{aligned}\int_0^{\pi/2} \vec{r}(t)\,\mathrm{d}t &= \int_0^{\pi/2} \langle 2\cos(t), \sin(t), 2t\rangle\,\mathrm{d}t \\ &= \left\langle 2\sin(t), -\cos(t), t^2\right\rangle\Big|_0^{\pi/2} \\ &= \boxed{\left\langle 2, 1, \frac{\pi^2}{4}\right\rangle}\end{aligned}$$

## Indefinite Integrals

What about indefinite integrals? Recall that the indefinite integral of a real-valued function is its most general antiderivative. The same is true of vector functions. Let's illustrate this with an example.

## Example

Evaluate $\int \langle 2\cos(t), \sin(t), 2t\rangle\,\mathrm{d}t$.

We have:

$$\begin{aligned}\int \langle 2\cos(t), \sin(t), 2t\rangle\,\mathrm{d}t &= \left\langle \int 2\cos(t)\,\mathrm{d}t, \int \sin(t)\,\mathrm{d}t, \int 2t\,\mathrm{d}t\right\rangle \\ &= \left\langle 2\sin(t) + C_1, -\cos(t) + C_2, t^2 + C_3\right\rangle \\ &= \left\langle 2\sin(t), -\cos(t), t^2\right\rangle + \langle C_1, C_2, C_3\rangle \\ &= \boxed{\left\langle 2\sin(t), -\cos(t), t^2\right\rangle + \vec{C}}\end{aligned}$$

## Table of Contents



## Exercises

1. Let $\vec{r}(t) = \left\langle \sqrt{t}, (2-t) \right\rangle$. Evaluate $\vec{r}\,'(1)$. Then find $\vec{T}(1)$, the unit tangent vector to $\vec{r}(t)$ at $t = 1$.
2. Let $\vec{r}(t) = \langle 2\cos(t), \sin(t), t \rangle$. Find a set of parametric equations for the line $L$ that lies tangent to the graph of $\vec{r}(t)$ at the point $(0, 1, \pi/2)$.
3. Evaluate $\int \left\langle \frac{1}{t+1}, \frac{1}{t^2+1}, \frac{t}{t^2+1} \right\rangle \mathrm{d}t$ and $\int_0^1 \left\langle \frac{1}{t+1}, \frac{1}{t^2+1}, \frac{t}{t^2+1} \right\rangle \mathrm{d}t$.

## Solutions

1. $\vec{r}\,'(1) = \left\langle \frac{1}{2}, -1 \right\rangle, \quad \vec{T}(1) = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$
2. [One possible answer:] A set of parametric equations for L is:

$$\begin{aligned} x &= -2t \\ y &= 1 \\ z &= t + \frac{\pi}{2} \end{aligned}$$

3. $\int \left\langle \frac{1}{t+1}, \frac{1}{t^2+1}, \frac{t}{t^2+1} \right\rangle \mathrm{d}t = \left\langle \ln(t+1), \arctan(t), \frac{1}{2}\ln(t^2+1) \right\rangle + \vec{C}$;
   and $\int_0^1 \left\langle \frac{1}{t+1}, \frac{1}{t^2+1}, \frac{t}{t^2+1} \right\rangle \mathrm{d}t = \left\langle \ln(2), \frac{\pi}{4}, \frac{1}{2}\ln(2) \right\rangle$.