12.6: Cylinders and Quadratic Surfaces

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Overview

In the previous section, we said that a *surface* in $\mathbb{R}^3$ can be thought of as a distorted sheet of paper. We then went on to give general equations for one class of surfaces: planes. Earlier in the chapter, you also studied the general equation for another class of surfaces: spheres! In fact, we will spend most of this course studying the calculus of surfaces, and thus we will need a library of examples to choose from, to help inform our discourse. This raises some elementary questions: what other classes of surfaces are there? What are their equations? What do they look like? How can we sketch them? These are the questions we will approach in this section.

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Definition and Example

You are, of course, already familiar with one class of cylinders: right circular cylinders; the standard can-shaped ones. In math, however, we think of cylinders as a general class of objects. Let's begin with a formal definition:

A **cylinder** is a surface made up of lines that are all parallel to a given line, passing through a given plane curve.

So, for example, a right circular cylinder is made up of vertical lines all parallel to each other, passing through a circle.

For more examples, see the board for the graphs of the cylinders C_1 and C_2 with equations $x^2 + y^2 = 1$ and $z = x^2$ respectively, and a more general description of how to sketch the graphs of cylinders.

Recognizing Cylinders: A Missing Variable

If you sit down to an equation of a surface, how can you tell if it's a cylinder? Here's a handy rule: if the equation is missing one or more of the variables x, y, or z, then the graph of the equation is a cylinder (Note: these are not the only equations whose graphs are cylinders! These are just one common type of cylinder you might encounter.)

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Utility

In our examples above, we got pretty lucky: we were able to quickly sketch the given cylinders because we knew what the planar versions of their graphs looked like.

But what if we were handed the equation of a surface at random, for example, $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$? How might we approach sketching this surface? As we progress in the course we will learn many techniques to try, but for now, **traces** provide one solid answer.

Definition

Traces are cross-sections of surfaces parallel to the coordinate planes. We obtain them by setting one of x, y, or z equal to a constant and sketching the resulting plane curve.

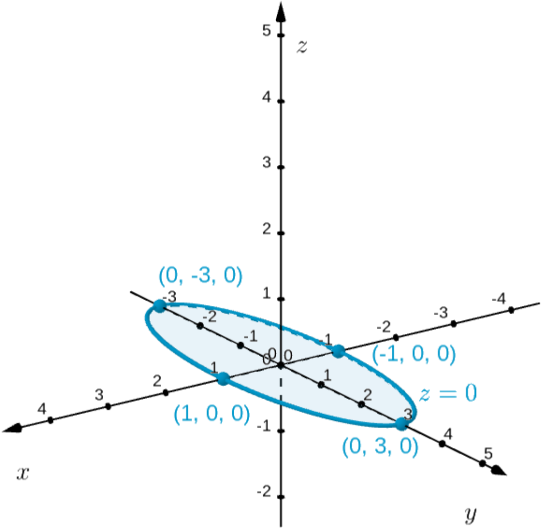
If we draw several traces of a surface, we can then patch them together to help sketch the surface itself.

Example

Sketch some traces of the surface S with equation $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$. Then use these traces to sketch the surface.

Let's start with cross-sections parallel to the xy-plane. In fact, let's start with the cross-section *in* the xy-plane by setting $z = 0$. The resulting cross-section of the surface is an ellipse about the origin in the xy coordinate plane: $x^2 + \frac{y^2}{9} = 1$. We plot this trace (this ellipse) in the xy-plane in $\mathbb{R}^3$ as on the following slide.

Example, cont.



Example

To help flesh out the surface even further, we'll try a couple of other simple traces: the trace in $y = 0$ and the trace in $x = 0$. Setting $y = 0$ in the equation of the surface S yields the trace

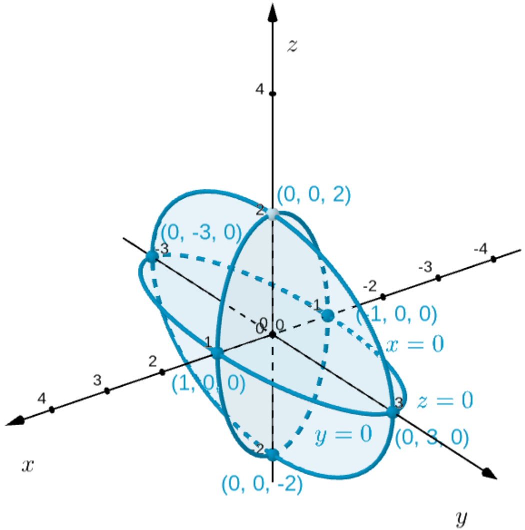
$$x^2 + \frac{z^2}{4} = 1$$

another ellipse; and setting $x = 0$ in the equation of the surface S yields the trace

$$\frac{y^2}{9} + \frac{z^2}{4} = 1$$

yet another ellipse! We again plot each of these traces in their respective locations in $\mathbb{R}^3$ as on the following slide.

Example, cont.



Example

You probably have a pretty good idea of what this surface looks like now!

Let's do one more trace, just to see how the process works when we set one of x, y, or z equal to a non-zero number. Let's try sketching the trace in $y = 2$, i.e. the cross-section of S at $y = 2$.

Setting $y = 2$ in the equation of the surface S yields the trace

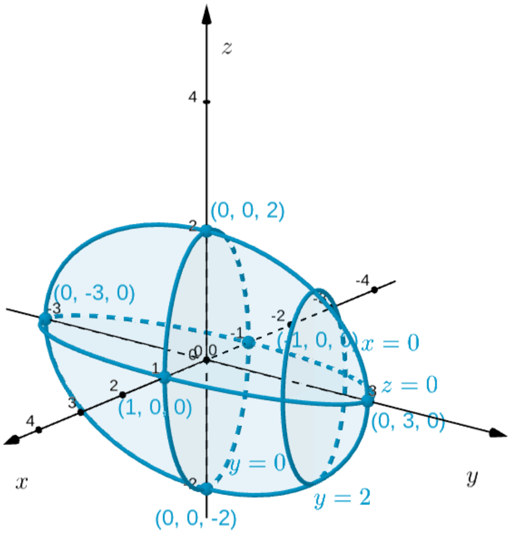
$$x^2 + \frac{4}{9} + \frac{z^2}{4} = 1$$

i.e.

$$x^2 + \frac{z^2}{4} = \frac{5}{9}$$

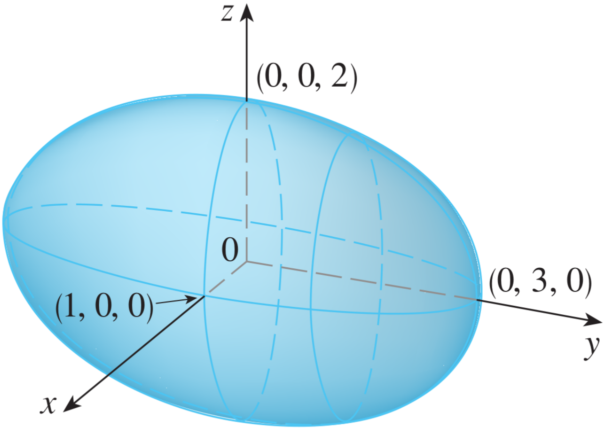
Again, this is an ellipse! We now plot this trace (this ellipse) in $\mathbb{R}^3$ inside the plane $y = 2$.

Example, cont.



Example, cont.

These cross-sections seem to be forming an egg-shaped surface, called an **ellipsoid**. Here's what it looks like as a nicely-rendered object (note that our traces are still visible!):



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Definition

There's one final family of surfaces that you should learn to sketch comparatively quickly: the quadratic surfaces.

A **quadratic surface** is the graph of a second-degree polynomial in x, y, and z. Its most general form is:

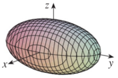
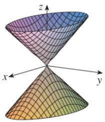
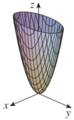
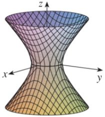
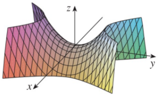
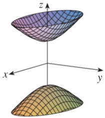
$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

for A through J constants.

Examples of quadratic surfaces include the unit sphere $x^2 + y^2 + z^2 = 1$, the ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$ from above, and the cylinder $x^2 + y^2 = 1$, also above, i.e., most of the surfaces we have studied so far (and, in fact, most of the surfaces we *will* study, because these are the simplest ones to work with, after planes).

Table of Quadratic Surfaces

Here are some common quadratic surfaces you should know:

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Example

Identify and sketch the quadratic surface S_1 which is given by the equation $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$.

Examining the chart above, we see that S_1 is a

$\boxed{\text{hyperboloid of one sheet}}$.

The trace of the figure in $z = 0$ (the xy-plane) is an ellipse through the points $(2, 0, 0)$ and $(0, 1, 0)$, so a rough sketch of the surface looks like this:

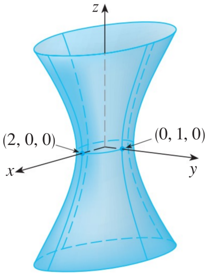


Figure: S_1

Example

Identify and sketch the surface S_2 which is given by the equation $4x^2 - y^2 + 2z^2 + 4 = 0$.

We should begin by putting the equation of S_2 in **standard form**, where the constant is both on the right and scaled to 1:

$$4x^2 - y^2 + 2z^2 = -4$$
$$\Rightarrow \quad -x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$$

With two negative signs on the left, we see that this is a $\boxed{\text{hyperboloid of two sheets}}$.

Example, cont.

Since y does not have a negative sign in our standard-form equation for S_2, it must be that the two sheets open up along the y-axis. Setting x and z equal to 0, we see that the sheets intersect the y-axis at $(0, \pm 2, 0)$. Thus, we arrive at the following rough sketch:

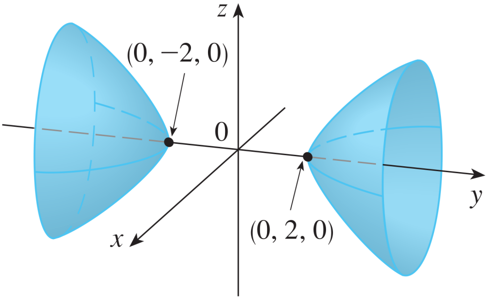


Figure: S_2

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Exercises

1. Sketch the surface $z = 1 - y^2$.
2. Use traces to sketch the surface $x = y^2 + 4z^2$.
3. Identify and sketch the quadratic surface $4x^2 - y + 2z^2 = 0$.
4. Identify and sketch the quadratic surface $x^2 + 2z^2 - 6x - y + 10 = 0$ [Hint: complete the square].

Solutions

Use a graphing utility of your choice to check your answers.