Estimating Sums

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1. Given positive real numbers $x_1, x_2, \ldots, x_n$ whose sum is an integer, prove that one can choose a nonempty proper sublist of the x_i such that the fractional part of the sum of this sublist is at most $1/n$.

2. (IMO, 1987) Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all zero, such that $|a_i| \leq k-1$ for all i, and $|a_1 x_1 + a_2 x_2 + \cdots + a_n x_n| \leq (k-1)\sqrt{n}/(k^n - 1)$.

3. Given a set of n nonnegative numbers whose sum is 1, prove that there exist two disjoint subsets, not both empty, whose sums differ by at most $1/(2^n - 1)$. Is this bound optimal for every n?

4. Let n be an odd positive integer and let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be nonnegative real numbers satisfying $x_1 + \cdots + x_n = y_1 + \cdots + y_n$. Show that there exists a proper, nonempty subset $J \subset \{1, \ldots, n\}$ such that
$$\frac{n-1}{n+1} \sum_{j \in J} x_j \leq \sum_{j \in J} y_j \leq \frac{n+1}{n-1} \sum_{j \in J} x_j.$$

5. Fix c with $1 < c < 2$ and, for $x_1 < x_2 < \cdots < x_n$, call the (unordered) set $\{x_1, x_2, \ldots, x_n\}$ "biased" if there exist $1 \leq i, j \leq n-1$ such that $x_{i+1} - x_i > c(x_{j+1} - x_j)$. Suppose $s_1, s_2, \ldots$ are distinct real numbers and $0 \leq s_i \leq 1$ for all i. Prove that there are infinitely many n such that the set $\{s_1, s_2, \ldots, s_n\}$ is biased.

6. (Poland, 1998) For $i = 1, 2, \ldots, 7$, a_i and b_i are nonnegative numbers such that $a_i + b_i \leq 2$. Prove that there exist distinct indices i, j such that $|a_i - a_j| + |b_i - b_j| \leq 1$.

7. Let $n \geq 3$ be odd. Given numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ from the interval $[0, 1]$, show that there exist distinct indices i, j such that $0 \leq a_i b_j - b_i a_j \leq 2/(n-1)$.

8. (Hungary, 1997) We are given 111 unit vectors in the plane whose sum is zero. Show that there exist 55 of the vectors whose sum has length less than 1.

9. (IMO, 1997) Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $|x_1 + \cdots + x_n| = 1$ and $|x_i| \leq (n+1)/2$ for all i. Show that there exists a permutation (y_i) of (x_i) such that $|y_1 + 2y_2 + \cdots + ny_n| \leq (n+1)/2$.

10. (Spain, 1997, adapted) The real numbers $x_1, \ldots, x_n$ have a sum of 0. Prove that there exists an index i such that $x_i + x_{i+1} + \cdots + x_j \geq 0$ for all $i \leq j < i + n$, where the indices are defined modulo n.

11. (Austria-Poland, 1995) Let v_1, v_2, $\ldots$, v_{95} be three-dimensional vectors with all coordinates in the interval $[-1, 1]$. Show that among all vectors of the form $s_1 v_1 + s_2 v_2 + \cdots + s_{95} v_{95}$, where $s_i \in \{-1, 1\}$ for each i, there exists a vector (a, b, c) satisfying $a^2 + b^2 + c^2 \leq 48$. Can the bound of 48 be improved?

12. (Iran, 1999) Suppose that r_1, r_2, $\ldots$, r_n are real numbers. Prove that there exists $S \subset \{1, 2, \ldots, n\}$ such that $1 \leq |S \cap \{i, i+1, i+2\}| \leq 2$ for $1 \leq i \leq n-2$, and
$$\left| \sum_{i \in S} r_i \right| \geq \frac{1}{6} \sum_{i=1}^{n} |r_i|.$$

13. (USA, 1996) For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S. Given a set A of n positive numbers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned into n classes so that, in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

14. (Russia, 1997) 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 3/2 times any other group.

15. (Iran, 1999) Suppose that $-1 \leq x_1, \ldots, x_n \leq 1$ are real numbers such that $x_1 + \cdots + x_n = 0$. Prove that there exists a permutation σ such that, for every $1 \leq p \leq q \leq n$,
$$|x_{\sigma(p)} + x_{\sigma(p+1)} + \cdots + x_{\sigma(q-1)} + x_{\sigma(q)}| \leq 2 - \frac{1}{n}.$$

16. (Iran, 1996) For $S = \{x_1, \ldots, x_n\}$ a set of n real numbers, all at least 1, we count the number of reals of the form $\sum_{i=1}^{n} \epsilon_i x_i$, $\epsilon \in \{0, 1\}$ lying in an open interval I of length 1. Find the maximum value of this count over all I and S.

17. (Beatty's Theorem) If α and β are positive irrationals satisfying $1/\alpha + 1/\beta = 1$, show that every interval $(n, n+1)$, where n is a positive integer, contains exactly one integer multiple of either α or β.

18. (Putnam, 1994) Let (r_n) be a sequence of positive reals with limit 0. Let S be the set of all numbers expressible in the form $r_{i_1} + \cdots + r_{i_{1994}}$ for positive integers $i_1 < i_2 < \cdots < i_{1994}$. Prove that every interval (a, b) contains a subinterval (c, d) whose intersection with S is empty.

19. Let x_1, $\ldots$, x_n be arbitrary real numbers. Prove that the number of pairs $\{i, j\}$ satisfying $1 < |x_i - x_j| < 2$ does not exceed $n^2/4$.