Counting in two ways

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June 28, 2005

1. (a) Find the number of triangles and diagonals in a triangulation of a regular n-gon.

   (b) Same question, but for "quadrangulations"; which regular n-gons can be quadrangulated?

2. In each cell of a 5×5 square grid is written either $+1$ or -1. The product of the values in each row and each column is computed. Is it possible that the sum of these ten values is zero? Same problem for a 4×4 square grid.

3. Is it possible to place 10 numbers from $\{1, 2, \ldots, 15\}$ into the following circles such that the absolute values of the differences of pairs of adjacent circles are 1, 2, …, 14 in some order?

4. (Iran '96) The top and bottom edges of a chessboard are identified together, as are the left and right edges, yielding a torus. Find the maximum number of knights which can be placed so that no two attack each other.

5. There are n pieces of candy in a pile. One is allowed to separate a pile into two piles, and add the product of the sizes of the resulting piles to a running total. The process terminates when each piece of candy is in its own pile. Show that the final sum is independent of the sequence of operations performed.

6. Twenty-five people form several committees. Each committee has five members, and any two committees have at most one common member. Determine, with justification, the maximum number of committees.

7. Let M be a set with seven elements, and let $A_1, A_2, \ldots, A_7$ be subsets of M such that

   (i) each A_i has at least three elements;

   (ii) each pair of elements of M is contained in exactly one A_i.

   Prove that each pair of subsets A_i and A_j share exactly one common element.

8. (Russia '96) In the Duma there are 1600 delegates, who have formed 16000 committees of 80 people each. Prove that one can find two committees having no fewer than four common members.

9. (China '96) Eight singers participate in an art festival where m songs are performed. Each song is performed by 4 singers, and each pair of singers performs together in the same number of songs. Find the smallest m for which this is possible.

10. Let $a_1, a_2, \ldots, a_{100}$ and $b_1, b_2, \ldots, b_{100}$ be 200 distinct real numbers. Construct a 100×100 table with $a_i + b_j$ written in the ith row and jth column. Suppose that the product of the entries of each column is 1. Prove that the product of the entries of each row is -1.

11. At a meeting of $12k$ people, each person exchanges greetings with exactly $3k + 6$ others. For any two people, the number who exchange greetings with both is the same. How many people are at the meeting? (You must also show that your answer(s) can be realized.)

Problems on combinatorial sums

1. (a) Show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

   (b) Show that
$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

   (c) Show that
$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + n \cdot \binom{n}{n} = n \cdot 2^{n-1}.$$

2. Show that for any nonnegative integers n and k we have
$$\binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots = \binom{n+1}{k+1}.$$

3. Show that for any nonnegative integers a, b, k, we have
$$\binom{a+b}{k} = \sum_{i=0}^{k} \binom{a}{i}\binom{b}{k-i}.$$

4. Show that
$$\sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}.$$