Rational and Irrational Numbers

Putnam Practice

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A **rational number** is one that can be expressed in the form a/b, where a, b are integers and $b \neq 0$. To represent a given non-zero rational number, we can choose a/b such that a is an integer, b is a natural number, and $(a, b) = 1$. We shall say then that the representative fraction is in lowest terms. An easy consequence of the definition is that any rational number has a periodic decimal expansion. Real numbers with non-repeating decimal expansions cannot be expressed in the form a/b and are called **irrational**.

A number that satisfies an equation of the form

$$c_0x^n + c_1x^{n-1} + ... + c_{n-1}x + c_n = 0;$$

where $c_0, ..., c_n$ are integers and $c_0 \neq 0$ is called **algebraic**. A number that is not algebraic is called **transcendental**. It is known that π and e are transcendental numbers.

Theorem 1 (Rational Root Theorem) *If $c_0, c_1, ...c_n$ are integers, a/b is in lowest terms and $x = a/b$ is a root of the equation*

$$c_0x^n + c_1x^{n-1} + ... + c_{n-1}x + c_n = 0,$$

then $a|c_n$ and $b|c_0$.

Example 1 *Prove that $\sqrt{2} + \sqrt{3}$ is irrational.*

Solution: Set $x = \sqrt{2} + \sqrt{3}$. Squaring, we find $x^2 = 5 + 2\sqrt{6}$ and $(x^2 - 5)^2 = 24$. Thus x is a root of equation $x^4 - 10x^2 + 1 = 0$. However, according to the Rational Root Theorem, the only possible rational roots of this equation are $x = 1, -1$. Neither 1 nor -1 satisfies the equation, so all its roots are irrational. In particular, $\sqrt{2} + \sqrt{3}$ is irrational.

Example 2 (1974 Putnam) *Prove that if α is a real number such that $\cos(\alpha\pi) = 1/3$ then α is irrational.*

Solution: From the addition formula for the cosine we have

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta.$$

Set $x = 2\cos\theta$ and $P_n(x) = 2\cos n\theta$, then

$$P_{n+1}(x) = xP_n(x) - P_{n-1}(x),$$

with $P_0(x) = 2$ and $P_1(x) = x$. By induction we can see that $P_n(x)$ is a monic polynomial. Suppose $\cos(m\pi/n) = a/b$ where a, b, m, n are integers. Then

$$P_n(2a/b) = 2\cos(m\pi) = 2(-1)^m,$$

so $x = 2a/b$ is a root of the equation

$$x^n + ... + c_{n-1}x + c_n = 0.$$

By Rational Root Theorem, $2a/b = 2\cos(m\pi/n)$ must be an integer, and since $|\cos\theta| \leq 1$ for all θ, the only possibilities are $2a/b = 0, \pm1, \pm2$. Thus $0, \pm1/2, \pm1$ are the only possible rational values for $\cos(\alpha\pi)$ if α is rational. In particular, if $\cos(\alpha\pi) = 1/3$ then α is irrational.

Example 3 (1949 Putnam) *Let a/b (in lowest terms) represent a rational number that lies in the open interval $(0, 1)$. Prove that*

$$|a/b - \sqrt{2}/2| > \frac{1}{4b^2}.$$

Solution: Since $\sqrt{2}$ is irrational, we know that $2a^2 - b^2 \neq 0$. But $2a^2 - b^2$ is an integer, so in fact $|2a^2 - b^2| \geq 1$. It follows that

$$|(a/b - \sqrt{2}/2)(a/b + \sqrt{2}/2)| = \frac{|2a^2 - b^2|}{2b^2} \geq \frac{1}{2b^2}.$$

But a/b and $\sqrt{2}/2$ are each in $(0, 1)$, so

$$a/b + \sqrt{2}/2 < 2.$$

Consequently,

$$|a/b - \sqrt{2}/2| > \frac{1}{4b^2}.$$

1 Problems

1. In the process of expanding the rational number a/b, where $0 < b < 100$, a student obtained the block of digits 143 somewhere beyond the decimal point. Prove that he made a mistake.

2. Prove that each of the following numbers is irrational.

   - $\sqrt{2} + \sqrt[3]{3}$
   - $\log_{10} 2$
   - $\pi + \sqrt{2}$

   Hint: to prove the last item, show that if $x + \sqrt{2}$ is rational, then x is algebraic.

3. Show that $x = 2\cos(\pi/7)$ satisfies the equation

   $$x^3 + x^2 - 2x - 1 = 0.$$

   Using this fact, show that $\cos(\pi/7)$ is irrational.

4. Prove that there is no set of integers m, n, p except $0, 0, 0$ for which $m + n\sqrt{2} + p\sqrt{3} = 0$. [1955 Putnam]