Putnam Seminar 2003

More problems from “Mathematical Miniatures”.

Hermite’s identity

Hermite’s identity is

$$\lfloor nx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor,$$

where x denotes a real number and n a positive integer. (Can you prove it?)

1. (From the 1968 International Mathematics Olympiad) Prove that for all positive integers n,

$$\left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor + \cdots = n.$$

Somewhat similar to the preceding problem is:

2. Evaluate the difference between the numbers

$$\sum_{k=0}^{2000} \left\lfloor \frac{3^k + 2000}{3^{k+1}} \right\rfloor \qquad \text{and} \qquad \sum_{k=0}^{2000} \left\lfloor \frac{3^k - 2000}{3^{k+1}} \right\rfloor .$$

3. (From the 1981 USA Mathematical Olympiad) If x is a positive real number and n a positive integer, prove that

$$\lfloor nx \rfloor \geq \frac{\lfloor x \rfloor}{1} + \frac{\lfloor 2x \rfloor}{2} + \cdots + \frac{\lfloor nx \rfloor}{n}.$$