Pigeonhole Principle

Putnam practice

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If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

1 Examples

1. Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

2. Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

3. (Problem 12) Prove that for any real number α and any positive integer n there exist integers p, q with $1 \leq q \leq n$ such that
$$|\alpha - p/q| < \frac{1}{qn}$$

4. Prove that every sequence of $(m-1)(n-1)+1$ distinct real numbers has either an increasing subsequence with m terms or a decreasing subsequence with n terms.

2 Previous Problems

1. Placing 5 points in a unit square, prove that at least two points are no more then $1/\sqrt{2}$ distance apart.

2. If $a_1, a_2, ..., a_n$ are any integers, show that there exist indices i and j such that $a_i + a_{i+1} + ... + a_j$ is divisible by n.

3. Let A be any set of 19 integers chosen from the arithmetic progression $1, 4, 7, ..., 100$. Prove that there exist two integers in A whose sum is 104.

4. Suppose that there are 101 points in the plane with the property that, of any three, some two are less than 1 unit apart. Show that there exists a circle of radius 1 containing at least 51 points.

3 More Problems

1. During a month with 30 days a baseball team plays at least 1 game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

2. (Putnam 2000) Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.