Putnam Seminar 2004

More problems from "Mathematical Miniatures".

1. At n distinct points of a circular race course, n cars are ready to start. Each of them covers the course in an hour. At a given signal every car selects one of the two possible directions and starts immediately. Whenever two cars meet, both of them change directions and go on without loss of speed. Show that at a certain moment each car will be at its starting point.

2. (from the 1973 Moscow Olympiad) Twelve painters live in 12 red and blue houses built around a circular lane. Each month one of them goes counterclockwise along the lane and, starting from his own house, repaints the houses according to the following rule: If a house is red, he paints it blue and passes to the next house; if a house is blue, he paints it red and goes home. Each painter does this once a year. Prove that if at least one of the houses is red, then a year later each house will have its initial color.

3. (from the 1995 USAMO) Let p be an odd prime. The sequence $(a_n)_{n\geq 0}$ is defined as follows: $a_0 = 0$, $a_1 = 1$, $\dots$, $a_{p-2} = p-2$ and, for all $n \geq p-1$, a_n is the least integer greater than a_{n-1} that does not form an arithmetic progression of length p with any of the preceding terms. Prove that, for all n, a_n is the number obtained by writing n in base $p-1$ and reading it in base p.