Progressions and Sums

Putnam Practice

October 6, 2004

An **arithmetic progression** is a sequence of numbers $\{a_k\}$ with $a_2 - a_1 = a_3 - a_2 = ... = d$. Note

$$a_k = \frac{a_{k-1} + a_{k+1}}{2} \text{ and } a_k = a_1 + (k-1)d$$

A sequence $\{b_k\}$ is a **geometric progression** if there is a number r such that $b_{k+1} = rb_k$. The sum of a finite geometric progression is

$$S_n = \frac{b_1(1 - r^n)}{1 - r}$$

if $r \neq 1$, otherwise it is nb_1.

Denote by $S_r(n) = \sum_{k=1}^{n} k^r$, then

$$S_1(n) = n(n+1)/2$$

$$S_2(n) = n(n+1)(2n+1)/6$$

$$S_3(n) = (n(n+1)/2)^2$$

$$S_r(n) = \frac{B_{r+1}(n+1) - B_{r+1}(0)}{r+1}$$

where $B_m(x)$ is the Bernoulli polynomial of degree m. They satisfy $B_m(x+1) - B_m(x) = mx^{m-1}$.

1 Problems

1. Let the sequence of natural numbers be partitioned into groups as follows:

$$1, (2, 3), (4, 5, 6), (7, 8, 9, 10), ...$$

Find the sum of the integers in the n-th group.

2. Prove that the powers of 2 are the only positive integers that cannot be written as the sum of two or more consecutive positive integers.

3. (1979 Putnam) Let $x_1, x_2, x_3, ...$ be a sequence of non-zero real numbers satisfying
$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}$$
for $n = 3, 4, 5, ...$ Establish a necessary and sufficient condition on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

4. Show that
$$\sin(\pi/n) + \sin(2\pi/n) + ... + \sin((n-1)\pi/n) = \cot(\pi/(2n))$$

5. (1972 Putnam) Show that there are no four consecutive binomial coefficients
$$\binom{n}{k}, \binom{n}{k+1}, \binom{n}{k+2}, \binom{n}{k+3}$$
(n,k are positive integers and $k + 3 \leq n$) which are in arithmetic progression.

6. Show that if a, b, c are positive numbers such that a^2, b^2, c^2 are in arithmetic progression, then
$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$$
are in arithmetic progression.