Putnam Seminar 2004

More problems from "Mathematical Miniatures". Here are three shorter problems:

1. The real numbers $x_1, x_2, \dots x_n$ satisfy the conditions

$$x_1 + x_2 + \cdots + x_n = 0, \qquad x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

Prove that the product of some pair of these numbers is less than or equal to $-1/n$.

2. The positive integers $x_1, x_2, \dots, x_{100}$ satisfy the equation

$$\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_{100}}} = 20.$$

Prove that at least two of them are equal.

3. It is well known that the divisibility tests for division by 3 and 9 do not depend on the order of the decimal digits. Prove that 3 and 9 are the only positive integers with this property. More precisely, if an integer $d > 1$ has the property that $d|n$ implies $d|n_1$, where n_1 is obtained from n through an arbitrary permutation of its digits, then $d = 3$ or $d = 9$.

You may have seen problems before whose solution involved the pigeonhole principle. Here are a couple more:

4. (from the 1983 Dutch Olympiad) Suppose 111 points are given within an equilateral triangle of side 15. Prove that it is always possible to cover at least 3 of these points by a round coin of diameter $\sqrt{3}$ (part of which may lie outside the triangle).

5. (from the 1980 Moscow Olympiad) Several arcs of great circles are located on a unit sphere. The sum of their lengths is less than π. Prove that there exists a plane through the center of the sphere that intersects none of these arcs.