Putnam Seminar 2003

Here are a few problems from "Mathematical Miniatures" by S. Savchev and T. Andreescu.

A telescoping sum

1. Evaluate in closed form

$$\sum_{k=1}^{n} \frac{k}{(k+1)!}.$$

2. Compute the sum

$$\sum_{k=1}^{n} \frac{k+1}{(k-1)!+k!+(k+1)!}.$$

3. (from the 1986 Polish Olympiad) Prove that for each $n \geq 3$, the number $n!$ can be represented as the sum of n distinct divisors of itself.

Lagrange's identity

For the next three problems, it is helpful to know *Lagrange's identity*, which states that

$$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2.$$

4. Let m and n be distinct positive integers. Represent m^6+n^6 as the sum of two perfect squares different from m^6 and n^6.

5. Let $P(x)$ be a polynomial with real coefficients so that $P(x) \geq 0$ for all real x. Prove that there exist polynomials with real coefficients, $Q_1(x)$ and $Q_2(x)$, such that

$$P(x) = Q_1^2(x) + Q_2^2(x) \qquad \text{for all } x.$$

6. (from the 1985 British Olympiad) Show that the equation $x^2+y^2 = z^5+z$ has infinitely many relatively prime integer solutions.