Polynomials

Putnam practice

Sept. 24, 2003

1 Complex Numbers

Complex numbers can be written in the form $x + iy$, where x, y are real numbers and $i = \sqrt{-1}$. Their addition and multiplication are defined by

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

and

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

Every complex number other than 0 has a multiplicative inverse

$$(x + iy)^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

We denote the set of complex numbers by $\mathbb{C}$. With the operations of addition and multiplication as defined, $\mathbb{C}$ is a field.

We say x is a *real part* of z, write $x = Re(z)$, and y is an *imaginary part* of z, write $y = Im(z)$. The *complex conjugate* of $z = x + iy$ is the complex number $\bar{z} = x - iy$. The *modulus* of z is the nonnegative real number $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$.

The complex number $z = x + iy$ can be associated with the point (x, y) in the Cartesian plane. Then $|z|$ is the distance from z to the origin. By introducing polar coordinates (r, θ) we can write

$$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where $r = \sqrt{x^2 + y^2}$ and θ is chosen so that $x = r\cos\theta$ and $y = r\sin\theta$. Then r is the modulus of z and θ is the *argument* of z. The polar form is particularly convenient for multiplication. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Theorem 1 (De Moivre) *For every integral number n*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Using De Moivre's Theorem, we can develop an understanding of roots of complex numbers, starting with roots of unity. To say that $z = r(\cos\theta + i\sin\theta)$ is an n-th root of unity means that $z^n = 1$ and requires

$$r^n(\cos n\theta + i\sin n\theta) = 1.$$

To satisfy this requirement, we need $r = 1$ and $\cos n\theta = 1$. The latter is satisfied if and only if $\theta = 2\pi k/n$, where k is an integer. It follows that every n-th root of unity is of the form ϵ^k, where

$$\epsilon = \cos(2\pi/n) + i\sin(2\pi/n).$$

The set of n-th roots of unity has a beautiful description as a set of points in the complex plane. The roots are the vertices of a regular polygon with n sides inscribed in the circle $|z| = 1$.

2 Polynomials

Theorem 2 (Fundamental Theorem of Algebra) *Any polynomial with complex coefficients has a root in $\mathbb{C}$.*

Theorem 3 (Uniqueness Theorem) *If $P(x)$ and $Q(x)$ are polynomials, each of degree at most n, and $P(x_i) = Q(x_i)$, for $i = 1, ...m$, where $x_1, ...x_m$ are distinct complex numbers and $m > n$, then P and Q are identical.*

Theorem 4 (Division Algorithm) *If $F(x)$ and $G(x) \neq 0$ are polynomials, then there exist unique polynomials $Q(x)$ and $R(x)$ such that*

$$F(x) = Q(x)G(x) + R(x)$$

where either $R(x)$ is the zero polynomial or $\deg R(x) < \deg G(x)$.

Theorem 5 *Let $F(x)$ be a polynomial. When $F(x)$ is divided by $x - a$, the remainder is $F(a)$. Thus $x = a$ is a root of $F(x) = 0$ if and only if $x - a$ is a factor of $F(x)$.*

3 Problems

1. Dividing $x^{2001} - 1$ by $(x^2+1)(x^2+x+1)$, what is the remainder?

2. If $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ are polynomials for which

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (1 + x + x^2 + x^3 + x^4)S(x)$$

   prove that $x - 1$ divides $P(x)$.

3. Determine all polynomials $P(x)$ such that $P(x^2+1) = P^2(x) + 1$ and $P(0) = 0$.

4. Find all nonconstant polynomials P with real coefficients such that $P(x^2) = P(x)P(x-1)$ for all x.

5. Factor $(a+b+c)^3 - (a^3+b^3+c^3)$.