1 Complex Numbers

Complex numbers can be written in the form \(x + iy\), where \(x, y\) are real numbers and \(i = \sqrt{-1}\). Their addition and multiplication are defined by

\[(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)\]

and

\[(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\]

Every complex number other than 0 has a multiplicative inverse

\[(x + iy)^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}\]

We denote the set of complex numbers by \(\mathbb{C}\). With the operations of addition and multiplication as defined, \(\mathbb{C}\) is a field.

We say \(x\) is a real part of \(z\), write \(x = \text{Re}(z)\), and \(y\) is an imaginary part of \(z\), write \(y = \text{Im}(z)\). The complex conjugate of \(z = x + iy\) is the complex number \(\bar{z} = x - iy\). The modulus of \(z\) is the nonnegative real number \(|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}\).

The complex number \(z = x + iy\) can be associated with the point \((x, y)\) in the Cartesian plane. Then \(|z|\) is the distance from \(z\) to the origin. By introducing polar coordinates \((r, \theta)\) we can write

\[x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}\]

where \(r = \sqrt{x^2 + y^2}\) and \(\theta\) is chosen so that \(x = r \cos \theta\) and \(y = r \sin \theta\). Then \(r\) is the modulus of \(z\) and \(\theta\) is the argument of \(z\). The polar form is particularly convenient for multiplication. If \(z_1 = r_1(\cos \theta_1 + i \sin \theta_1)\) and \(z_2 = r_2(\cos \theta_2 + i \sin \theta_2)\), then

\[z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))\]
Theorem 1 (De Moivre) For every integral number $n$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$ 

Using De Moivre’s Theorem, we can develop an understanding of roots of complex numbers, starting with roots of unity. To say that $z = r(\cos \theta + i \sin \theta)$ is an $n$-th root of unity means that $z^n = 1$ and requires

$$r^n(\cos n\theta + i \sin n\theta) = 1.$$ 

To satisfy this requirement, we need $r = 1$ and $\cos n\theta = 1$. The latter is satisfied if and only if $\theta = 2\pi k/n$, where $k$ is an integer. It follows that every $n$-th root of unity is of the form $\epsilon^k$, where

$$\epsilon = \cos(2\pi/n) + i\sin(2\pi/n).$$

The set of $n$-th roots of unity has a beautiful description as a set of points in the complex plane. The roots are the vertices of a regular polygon with $n$ sides inscribed in the circle $|z| = 1$.

2 Polynomials

Theorem 2 (Fundamental Theorem of Algebra) Any polynomial with complex coefficients has a root in $\mathbb{C}$.

Theorem 3 (Uniqueness Theorem) If $P(x)$ and $Q(x)$ are polynomials, each of degree at most $n$, and $P(x_i) = Q(x_i)$, for $i = 1, \ldots, m$, where $x_1, \ldots, x_m$ are distinct complex numbers and $m > n$, then $P$ and $Q$ are identical.

Theorem 4 (Division Algorithm) If $F(x)$ and $G(x) \neq 0$ are polynomials, then there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$F(x) = Q(x)G(x) + R(x)$$

where either $R(x)$ is the zero polynomial or $\deg R(x) < \deg G(x)$.

Theorem 5 Let $F(x)$ be a polynomial. When $F(x)$ is divided by $x - a$, the remainder is $F(a)$. Thus $x = a$ is a root of $F(x) = 0$ if and only if $x - a$ is a factor of $F(x)$. 

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3 Problems

1. Dividing $x^{2001} - 1$ by $(x^2 + 1)(x^2 + x + 1)$, what is the remainder?

2. If $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ are polynomials for which

$$P(x^5) + xQ(x^5) + x^2 R(x^5) = (1 + x + x^2 + x^3 + x^4)S(x)$$

prove that $x - 1$ divides $P(x)$.

3. Determine all polynomials $P(x)$ such that $P(x^2 + 1) = P^2(x) + 1$ and $P(0) = 0$.

4. Find all nonconstant polynomials $P$ with real coefficients such that $P(x^2) = P(x)P(x - 1)$ for all $x$.

5. Factor $(a + b + c)^3 - (a^3 + b^3 + c^3)$. 