A famous example of a Diophantine equation is Pell equation. It is an equation of the form

\[ x^2 - Dy^2 = 1 \]

with \( D \) a positive integer that is not a perfect square. To find all positive integer solutions of this equation, one first determines a minimal solution (i.e. the solution \((x_0, y_0)\) for which \(x_0 + y_0 \sqrt{D}\) is minimal). There is a general way to compute this minimal solution, however, in all problems below the minimal solution is easy to guess.

The other solutions are given by

\[ x_n + y_n \sqrt{D} = (x_0 + y_0 \sqrt{D})^n \]

It is easy to see that this formula yields solutions of the equation (multiply \(x_n + y_n \sqrt{D}\) by its conjugate \(x_n - y_n \sqrt{D} = (x_0 - y_0 \sqrt{D})^n\)). It is also easy to see that there are no other solutions.

For a generalized Pell equation

\[ ax^2 - by^2 = c \]

where \(a, b\) are not divisible by any square the solutions might not exist. If \(c = 1, a, b \neq 1\) and the minimal solution \((x_0, y_0)\) exists, then the general solution is generated by

\[ x_n \sqrt{a} + y_n \sqrt{b} = (x_0 \sqrt{a} + y_0 \sqrt{b})^{2n+1} \]

**Example 1** Prove that \(n^2 + (n + 1)^2\) is a perfect square for infinitely many natural numbers.

**Solution:** Write \(n^2 + (n + 1)^2 = m^2\) as

\[(2n + 1)^2 - 2m^2 = -1\]
Thus values of $n$ for which $n^2 + (n + 1)^2$ is a perfect square correspond to solutions of the Pell equation $x^2 - 2y^2 = -1$, where $x = 2n + 1$. One solution is $x = y = 1$. Let

$$x_k + y_k \sqrt{2} = (1 + \sqrt{2})^{2k+1}, \quad k \geq 2$$

It is easy to check these are solutions of $x^2 - 2y^2 = -1$ as well. Thus $n_k = \frac{x_k - 1}{2}$ (note $x_k$ is always odd). The first 3 values for $n$ are 3, 20, 119.

1 Problems

1. Find all natural numbers of the form $m(m + 1)/3$ that are perfect squares.

2. Solve the equation $(x + 1)^3 - x^3 = y^2$ in positive integers.

3. Find all positive integers $n$ for which both $2n+1$ and $3n+1$ are perfect squares.

4. Let $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$, and for positive integers $n$, define $d_n$ as the greatest common divisor of the entries of $A^n - I$, where $I$ is the identity matrix. Prove that $d_n \to \infty$ as $n \to \infty$.

5. Prove that there exist infinitely many positive integers $n$ such that $n^2 + 1$ divides $n!$. 

2