The Classical Inequalities

Putnam Practice

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**Arithmetic mean - Geometric mean inequality** says that for any \(n\) non-negative real numbers \(a_1, \ldots, a_n\) we have:

\[
\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}
\]

and equality holds \(\iff a_1 = a_2 = \ldots = a_n\). Define the **Harmonic mean** by

\[
H = \frac{n}{1/a_1 + 1/a_2 + \ldots + 1/a_n}
\]

then it is not hard to check that

\[
HM \leq GM \leq AM.
\]

**Theorem 1 (Power mean inequality)** Let \(a_1, \ldots, a_n\) be positive real numbers, and let \(\alpha\) be real. Let

\[
M_\alpha(a_1, \ldots, a_n) = \left(\frac{a_1^\alpha + \ldots + a_n^\alpha}{n}\right)^{1/\alpha}, \alpha \neq 0
\]

and

\[
M_0 = \sqrt[n]{a_1 \ldots a_n}.
\]

Then \(M_\alpha\) is an increasing function of \(\alpha\) unless \(a_1 = a_2 = \ldots = a_n\) (in which case \(M_\alpha\) is constant).

**Theorem 2 (Cauchy inequality)** For arbitrary real numbers \(a_1, a_2, \ldots, a_n\) and \(b_1, b_2, \ldots, b_n\) we have

\[
(a_1 b_1 + \ldots + a_n b_n)^2 \leq (a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2)
\]

Furthermore, equality holds if and only if there are two numbers \(\lambda, \mu\) not both zero, such that \(\lambda a_i = \mu b_i\) for all \(i\).
Let \( x = (x_1, ..., x_n) \in \mathbb{R}^n \) and norm of \( x \) be
\[
\|x\| = (x_1^2 + ... + x_n^2)^{1/2}.
\]

**Theorem 3** For any two elements \( x, y \in \mathbb{R}^n \)
\[
\|x + y\| \leq \|x\| + \|y\|.
\]
Equality holds if and only if there exist \( \lambda, \mu \) not both zero such that \( \lambda x = \mu y \).

**Problems:**

1. Show that if \( a, b, c \) are non-negative numbers, then
\[
(a + b)(b + c)(c + a) \geq 8abc
\]

2. (1975 Putnam) Let \( H_n = 1 + 1/2 + ... + 1/n \). Show that
\[
n(n + 1)^{1/n} < n + H_n
\]
for every \( n > 1 \)

3. Show that if \( a, b, c \) are positive real numbers, then
\[
\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}
\]

4. Show that if \( a, b > 0 \) and \( a + b = 1 \), then
\[
(a + 1/a)^2 + (b + 1/b)^2 \geq 25/2
\]

5. Let \( a, b, c \) be positive real numbers. Prove that
\[
1/a + 1/b + 1/c \leq \frac{a^8 + b^8 + c^8}{a^3b^3c^3}
\]

6. Prove Cauchy Inequality.
   Hint: 1) \( b_i = 0 \), true
   2) \( b_1^2 + ... b_n^2 > 0 \). Consider the polynomial \( P(x) = (a_1 - b_1x)^2 + (a_2 - b_2x)^2 + ... + (a_n - b_nx)^2 \). Note \( P(x) \geq 0 \) for any \( x \), in particular for
   \[
x = \frac{a_1b_1 + ... + a_nb_n}{b_1^2 + ... + b_n^2}
   \]