Putnam Seminar — Week 2

7. (B2, 2005) Find all positive integers n, $k_1, \dots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

8. Let n be a fixed positive integer and let $b(n)$ be the minimum value of $k + \dfrac{n}{k}$ as k is allowed to range through all positive integers. Prove that $b(n)$ and $\sqrt{4n+1}$ have the same integer part. (The "integer part" of a real number is the greatest integer which does not exceed it; e.g., for π it is 3, for $\sqrt{21}$ it is 4, for 5 it is 5.)

9. How many zeros does the function $f(x) = 2^x - 1 - x^2$ have on the real line? (I.e., how many real numbers x are there such that $2^x - 1 - x^2 = 0$?)

10. Evaluate

$$\lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n}.$$

11. (B4, 2005) For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \dots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \le m$. Show that $f(m,n) = f(n,m)$.

12. (B2, 2001) Find all pairs of real numbers (x,y) satisfying the system of equations

$$\begin{aligned} \frac{1}{x} + \frac{1}{2y} &= (x^2 + 3y^2)(3x^2 + y^2) \\ \frac{1}{x} - \frac{1}{2y} &= 2(y^4 - x^4). \end{aligned}$$