Putnam Seminar — Week 1

1. (A1, 2013) Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

2. (B1, 2013) For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

3. (A2, 2012) Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$

4. (A2, 2000) Prove that there exist infinitely many integers n such that n, $n+1$, $n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

5. (B2, 2004) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\,\frac{n!}{n^n}.$$

6. On the domain $0 \le \theta \le 2\pi$:

a) Prove that $\sin^2(\theta)\sin(2\theta)$ takes its maximum at $\pi/3$ and $4\pi/3$ (and hence its minimum at $2\pi/3$ and $5\pi/3$).

b) Show that

$$\left|\sin^2\theta\left\{\sin^3(2\theta)\sin^3(4\theta)\cdots\sin^3(2^{n-1}\theta)\right\}\sin(2^n\theta)\right|$$

takes its maximum at $\theta = \pi/3$ (this maximum may also be attained at other points).

c) Derive the inequality

$$\sin^2(\theta)\sin^2(2\theta)\sin^2(4\theta)\cdots\sin^2(2^n\theta) \le (3/4)^n.$$