EXISTENCE AND STABILITY OF A TWO-PARAMETER FAMILY OF SOLITARY WAVES FOR AN NLS-KDV SYSTEM

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Abstract. We prove existence and stability results for a two-parameter family of solitary-wave solutions to a system in which an equation of nonlinear Schrödinger type is coupled to an equation of Korteweg-de Vries type. Such systems model interactions between short and long dispersive waves. The results extend earlier results of Angulo, Albert and Angulo, and Chen. Our proof involves the characterization of solitary-wave solutions as minimizers of an energy functional subject to two constraints. To establish the precompactness of minimizing sequences via concentrated compactness, we establish the sub-additivity of the problem with respect to both constraint variables jointly.

1. Introduction

Both the nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^q u = 0 \tag{1.1}$$

for a complex-valued function u of $x \in \mathbb{R}$ and time t, and the generalized Korteweg-de Vries equation

$$v_t + v_{xxx} + v^p v_x = 0, \tag{1.2}$$

for a real-valued function v of x and t, are universal models for nonlinear waves in dispersive media. Equation (1.2) arises generically as a model for waves whose motion, to first order, is governed by the linear wave equation $v_t + v_x = 0$, but which on account of their long wavelength and small but finite amplitude are influenced by weak nonlinear and dispersive effects. Equation (1.2), on the other hand, describes the amplitude and phase modulations of long-wavelength, small-amplitude perturbations of a monochromatic short wave in a dispersive medium. Discussions of the canonical nature of these equations may be found, for example, in chapters 13 and 17 of [23], chapter 2 of [20], or chapter 10 of [19].

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In this paper we will consider a system describing the interaction of a nonlinear Schrödinger-type wave with a Korteweg-de Vries type wave:

$$\begin{aligned} iu_t + u_{xx} + \tau_1 |u|^q u &= -\alpha u v \\ v_t + v_{xxx} + \tau_2 v^p v_x &= -\frac{\alpha}{2}(|u|^2)_x, \end{aligned} \tag{1.3}$$

where τ_1, τ_2, and α are real constants. The form of the coupling terms in system (1.3) is also universal: the system arises as a model for interactions between long waves and long-wavelength envelopes of short waves in a variety of physical settings. For example, it appears in [14] and [15] as a model for the interaction between long gravity waves and capillary waves on the surface of shallow water. A system of similar form, but with the term v_{xxx} in the second equation replaced by $-v_{xxx}$, appears in [4] (see also [21]) as a model for the interaction of Langmuir waves and ion-acoustic waves in a plasma. The status of (1.3) as a generic model may be related to the fact that it has a Hamiltonian structure in which the Hamiltonian (the functional $E(u,v)$ defined below) has the coupling term $\alpha v|u|^2$. If one requires the coupling term to be a power series in $|u|^2$ and v, this is the simplest possible coupling one could expect.

We consider here the initial-value problem for (1.3) on the line, for (u,v) in the space $Y = H^1_{\mathbb{C}}(\mathbb{R}) \times H^1(\mathbb{R})$. (Here $H^1(\mathbb{R})$ and $H^1_{\mathbb{C}}(\mathbb{R})$ are L^2-based Sobolev spaces of real- and complex-valued functions on the line, respectively. For more details on our notation, see below.) In the case when $p = 1$ and $q = 2$, for arbitrary values of τ_1, τ_2, and α, this problem has been shown to be well-posed locally in time by Bekiranov et al. in [5], and global well-posedness was proved by Corcho and Linares [9]. Dias, Figueira and Oliveira [12] extended the global well-posedness result to the case when $p = 1$ and $1 < q < 4$, and their proof will work for $q = 1$ as well[1]. These results depend on the fact that the following functionals are conserved under the flow of (1.3):

$$E(u,v) = \int_{-\infty}^{\infty} \left(|u_x|^2 + v_x^2 - \beta_1 |u|^{q+2} - \beta_2 v^{p+2} - \alpha |u|^2 v \right) \, dx, \tag{1.4}$$

where $\beta_1 = 2\tau_1/(q+2)$ and $\beta_2 = 2\tau_2/((p+1)(p+2))$,

$$G(u,v) = \int_{-\infty}^{\infty} v^2 \, dx + \operatorname{Im} \int_{-\infty}^{\infty} u \overline{u}_x \, dx, \tag{1.5}$$

where $\overline{u}_x$ is the complex conjugate of u_x and $\operatorname{Im}(z)$ denotes the imaginary part of z, and

$$H(u) = \int_{-\infty}^{\infty} |u|^2 \, dx. \tag{1.6}$$

In other words, if (u,v) is a solution of (1.3) in Y, then $E(u,v)$, $G(u,v)$, and $H(u)$ are independent of time.

[1] João Paulo Dias, personal communication.

The methods used in this paper require the assumption that τ_1, τ_2 and α are positive, or at least non-negative. Also, in order for the term v^p in (1.3) or (1.4) to be defined when $v < 0$, we will assume in what follows that p is a positive rational number with odd denominator. Much of what is proved below should be readily extendable to versions of (1.3) with other nonlinearities, such as when $v^p v_x$ is replaced in (1.3) by $(|v|^p)_x$, in which case the analogue of Theorem 1.1 will hold for all real values of p such that $1 \leq p < 4$.

The purpose of this paper is to prove existence and stability results for (coupled) solitary traveling-wave solutions of (1.3). Such a solution is of the form

$$(u(x,t), v(x,t)) = \left(e^{i\omega t} e^{ic(x-ct)/2}\phi(x-ct), \psi(x-ct)\right), \tag{1.7}$$

where $c > 0$, $\omega \in \mathbb{R}$, and $\phi : \mathbb{R} \to \mathbb{C}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are functions that vanish at infinity, in the sense that $\phi \in H^1_{\mathbb{C}}$ and $\psi \in H^1$. Inserting the ansatz (1.7) into (1.3), we see that (u, v) is a solution of (1.3) if and only if ϕ and ψ satisfy the system of ordinary differential equations

$$\begin{aligned} -\phi'' + \sigma\phi &= \tau_1 |\phi|^q \phi + \alpha\phi\psi \\ -\psi'' + c\psi &= \frac{\tau_2}{p+1}\psi^{p+1} + \frac{\alpha}{2}|\phi|^2, \end{aligned} \tag{1.8}$$

where $\sigma = \omega - c^2/4$, and primes denote derivatives of a function of a single variable.

One question we address below is whether nontrivial solutions of (1.8) exist. Our existence result is obtained by studying the variational problem of finding, for given positive values of s and t, minimizers of $E(u, v)$ subject to the constraints that $\int_{-\infty}^{\infty} |u|^2\, dx = s$ and $\int_{-\infty}^{\infty} v^2\, dx = t$. The connection to solitary waves is due to the fact that equations (1.8) are the Euler-Lagrange equations for this variational problem, with σ and c playing the role of Lagrange multipliers. In Section 2, we use the method of concentration compactness to prove the relative compactness of minimizing sequences for the variational problem, and hence the existence of minimizers. This requires proving the strict subadditivity (see Lemma 2.12 below) of the function $I(s, t)$ defined for $s > 0$ and $t > 0$ by

$$I(s,t) = \inf\left\{E(f,g) : (f,g) \in Y, \int_{-\infty}^{\infty} |f|^2\, dx = s, \text{ and } \int_{-\infty}^{\infty} g^2\, dx = t\right\}. \tag{1.9}$$

For equations (1.1) or (1.2), the variational problems which characterize solitary waves depend on a single constraint parameter, and proofs of strict subadditivity are accomplished by simple arguments, dating back to Lions' original paper [17], which take advantage of homogeneities present in the equation. To prove strict subadditivity for the two-parameter problem defined in (1.9), however, seems to be more difficult. In [2], which treats the case where $p = 1$ and $\tau_1 = 0$, it was noted that strict subadditivity, as defined below in Lemma 2.12, holds for $\alpha = 1/6$ (corresponding to setting the parameter q in [2] equal to 2), and it was shown that strict subadditivity continues to hold for α in some neighborhood of 1/6. Here we are able to extend this result to all positive values of α, all non-negative values of τ_1, all positive valued of τ_2, all $p \in [1,4)$, and all $q \in [1,4)$. To do so, we rely on an argument due to Byeon [6] and Garrisi [11], which exploits the fact that the H^1 norms of certain functions are strictly decreased when the mass of the function is rearranged by symmetrization.

Before stating our existence result, let us define a minimizing sequence for $I(s,t)$ to be a sequence (f_n, g_n) in Y such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n|^2 = s, \ \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n^2 = t, \text{ and } \lim_{n\to\infty} E(f_n, g_n) = I(s,t).$$

Our existence result is the following.

Theorem 1.1. *Suppose $\alpha > 0$, $\tau_1 \geq 0$, $\tau_2 > 0$, $1 \leq q < 4$, and $1 \leq p < 4$, where p is a rational number with odd denominator. For $s > 0$ and $t > 0$, define*

(1.10)

$$\mathcal{S}_{s,t} = \left\{ (\phi, \psi) \in Y : E(\phi, \psi) = I(s,t), \int_{-\infty}^{\infty} |\phi|^2 \, dx = s, \text{and} \int_{-\infty}^{\infty} \psi^2 \, dx = t \right\}.$$

Then the following statements are true for all $s > 0$ and $t > 0$.

(i) The infimum $I(s,t)$ defined in (1.9) is finite.

(ii) Every minimizing sequence $\{(f_n, g_n)\}$ for $I(s,t)$ is relatively compact in Y up to translations. That is, there exists a subsequence $\{(f_{n_k}, g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that $\{(f_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k)\}$ converges strongly in Y to some (ϕ, ψ) in $\mathcal{S}_{s,t}$. In particular, the set $\mathcal{S}_{s,t}$ is non-empty.

(iii) Each function $(\phi, \psi) \in \mathcal{S}_{s,t}$ is a solution of (1.8) for some σ and c, and therefore when substituted into (1.7) yields a solitary-wave solution of (1.3).

(iv) For every (ϕ, ψ) in $\mathcal{S}_{s,t}$, we have that $\psi(x) > 0$ for all $x \in \mathbb{R}$, and there exist a number $\theta \in \mathbb{R}$ and a function $\tilde{\phi}$ such that $\tilde{\phi}(x) > 0$ for all $x \in \mathbb{R}$, and $\phi(x) = e^{i\theta}\tilde{\phi}(x)$. Also, the functions ψ and ϕ are infinitely differentiable on $\mathbb{R}$.

Notice that it is obvious from the definition of the sets $\mathcal{S}_{s,t}$ that they form a true two-parameter family, in that $\mathcal{S}_{s_1,t_1}$ and $\mathcal{S}_{s_2,t_2}$ are disjoint if $(s_1, t_1) \neq (s_2, t_2)$. Previously, Dias et al. [12] had proved that for $p \in \{1, 2, 3\}$ (with $\alpha > 3$ if $p = 1$), (1.3) has an infinite family of positive bound states which decay exponentially at infinity. Compared to the result of [12], ours has the advantages that we do not require $\alpha > 3$ when $p = 1$, and also that we obtain a true two-parameter family of solitary waves. In [12], nonempty sets $\mathcal{T}_{\delta,\mu}$ of solitary waves are obtained by minimizing E subject to $\int |u|^2 + \delta v^2 = \mu$, but it is not clear whether $\mathcal{T}_{\delta_1,\mu_1}$ is necessarily disjoint from $\mathcal{T}_{\delta_2,\mu_2}$ if $(\delta_1, \mu_1) \neq (\delta_2, \mu_2)$.

A separate question is that of stability of the solutions of (1.8) as solutions of the initial-value problem for (1.3). For $s > 0$ and $t \in \mathbb{R}$, define

$$W(s,t) = \inf\{E(h,g) : (h,g) \in Y,\ H(h) = s \text{ and } G(h,g) = t\}. \tag{1.11}$$

The variational problem associated to $W(s,t)$ is suitable for studying stability because not only the functional E being minimized, but also the constraint functionals G and H are conserved for (1.3). If minimizers (Φ,ψ) for $W(s,t)$ exist, they satisfy the Euler-Lagrange equations

$$\begin{aligned} -\Phi'' + \omega\Phi + ci\Phi' &= \tau_1|\Phi|^q\Phi + \alpha\Phi\psi \\ -\psi'' + c\psi &= \frac{\tau_2\psi^{p+1}}{p+1} + \frac{\alpha}{2}|\Phi|^2 \end{aligned} \tag{1.12}$$

where the real numbers c and ω are the Lagrange multipliers. These equations are satisfied by Φ and ψ if and only if the functions u and v defined by

$$(u(x,t), v(x,t)) = \left(e^{i\omega t}\Phi(x-ct), \psi(x-ct)\right) \tag{1.13}$$

are solutions of the NLS-KdV system (1.3). That is, solutions (Φ,ψ) of the variational problem for $W(s,t)$ are solitary-wave profiles, and (1.7) is recovered from (1.13) by setting $\Phi(x) = e^{icx/2}\phi(x)$.

We have the following stability result.

Theorem 1.2. *Suppose $\alpha > 0$, $\tau_1 \geq 0$, $\tau_2 > 0$, $1 \leq q < 4$, and $p = 1$. For $s > 0$ and $t \in \mathbb{R}$, define*

$$\mathcal{F}_{s,t} = \{(\Phi,\psi) \in Y : E(\Phi,\psi) = W(s,t), H(\Phi) = s, \text{ and } G(\Phi,\psi) = t\}\,. \tag{1.14}$$

Then the following statements are true for all $s > 0$ and $t \in \mathbb{R}$.

(i) The infimum $W(s,t)$ defined in (1.11) is finite.

(ii) Every minimizing sequence $\{(h_n, g_n)\}$ for $W(s,t)$ is relatively compact in Y up to translations. That is, if

$$\lim_{n\to\infty} H(h_n) = s,\ \lim_{n\to\infty} G(h_n,g_n) = t,\ \text{and}\ \lim_{n\to\infty} E(h_n,g_n) = W(s,t),$$

then there is a subsequence $\{(h_{n_k}, g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that $\{h_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k)\}$ converges strongly in Y to some $(\Phi,\psi) \in \mathcal{F}_{s,t}$. In particular, the set $\mathcal{F}_{s,t}$ is non-empty.

(iii) Each $(\Phi,\psi) \in \mathcal{F}_{s,t}$ is a solution of (1.12) for some ω and c, and therefore when substituted into (1.13) yields a solitary-wave solution of (1.3).

(iv) For every $(\Phi,\psi) \in \mathcal{F}_{s,t}$, let $a = \|\psi\|^2$ and $b = (t-a)/s$. Then there exist $\theta \in \mathbb{R}$ and a real-valued function $\tilde{\phi}$ such that $(\tilde{\phi},\psi) \in \mathcal{S}_{s,a}$ and

$$\Phi(x) = e^{i(-bx+\theta)}\tilde{\phi}(x) \tag{1.15}$$

on $\mathbb{R}$. Further, if $\tau_1 = 0$, then $a > 0$, $\psi(x) > 0$ for all $x \in \mathbb{R}$, and we can take $\tilde{\phi}$ to be everywhere positive on $\mathbb{R}$.

(v) The set $\mathcal{F}_{s,t}$ is a stable set of initial data for (1.3)*, in the following sense: for every $\epsilon > 0$, there exists $\delta > 0$ such that if $(h_0, g_0) \in Y$,*

$$\inf_{(\Phi,\psi)\in\mathcal{F}_{s,t}} \|(h_0, g_0) - (\Phi, \psi)\|_Y < \delta,$$

and $(u(x,t), v(x,t))$ is the solution of (1.3) *with*

$$(u(x,0), v(x,0)) = (h_0(x), g_0(x)),$$

then for all $t \geq 0$,

$$\inf_{(\Phi,\psi)\in\mathcal{F}_{s,t}} \|(u(\cdot,t), v(\cdot,t)) - (\Phi, \psi)\|_Y < \epsilon.$$

Furthermore, the sets $\mathcal{F}_{s,t}$ form a true two-parameter family, in that $\mathcal{F}_{s_1,t_1}$ and $\mathcal{F}_{s_2,t_2}$ are disjoint if $(s_1, t_1) \neq (s_2, t_2)$.

We remark that, if it is assumed that that (1.3) is globally well-posed in Y when $1 \leq p < 4/3$ (where p is rational with odd denominator), then the above stability result extends to these values of p as well, with the same proof.

From the definition of the variational problem for $W(s,t)$ it is clear that the sets $\mathcal{F}_{s,t}$ are invariant under the transformation

$$(\Phi(x), \psi(x)) \mapsto (e^{i\theta}\Phi(x-\xi), \psi(x-\xi)),$$

for every pair of real numbers θ and ξ, and so are at least two-dimensional in size. On the other hand, for a given solitary-wave profile (g,h) in $\mathcal{F}_{s,t}$, the orbit $\mathcal{O} = \{(u(x,t), v(x,t)) : t \in \mathbb{R}\}$ of the corresponding solitary wave is seen from (1.13) to be given by

$$\mathcal{O} = \left\{(e^{ict}\Phi(x-ct), \psi(x-ct)) : t \in \mathbb{R}\right\},$$

and hence is a proper (one-dimensional) subset of $\mathcal{F}_{s,t}$. Therefore Theorem 1.2 is somewhat weaker than an orbital stability result for the solitary waves in $\mathcal{F}_{s,t}$.

According to part (iv) of Theorem 1.2, in the case when $\tau_1 = 0$, each element (Φ, ψ) of $\mathcal{F}_{s,t}$ has a non-trivial second component ψ. However, we have not been able to establish this in the case when $\tau_1 > 0$. Even if $\mathcal{F}_{s,t}$ were to consist solely of solitary waves of the form $(\Phi, 0)$, however, it would still be of interest to know that $\mathcal{F}_{s,t}$ is stable in the sense described in part (v).

Theorem 1.2 generalizes the stability results of [8], which treated the case when $\tau_1 = 0$, $p = 1$, and $\alpha = 1/6$; and of [2], which treated the case when $\tau_1 = 0$, $p = 1$, and α is in some neighborhood of $1/6$. We also note the interesting paper of Angulo [3], which proves stability by a different method in the case when $\tau_1 = 0$, $p = 1$, $\alpha > 0$, and the wavespeed σ appearing in (1.8) is sufficiently small.

The remainder of the paper is organized as follows. In Section 2, after a number of preparatory lemmas, including Byeon and Garrisi's rearrangement lemma, we prove assertions (i) through (iv) of Theorem 1.1. Section 3 also begins with some preparatory lemmas, and then concludes with a proof of assertions (i) through (v) of Theorem 1.2.

Notation. For $1 \le p \le \infty$, we denote by L^p the space of all measurable functions f on $\mathbb{R}$ for which the norm $|f|_p$ is finite, where

$$|f|_p = \left(\int_{-\infty}^{\infty} |f|^p \, dx \right)^{1/p} \quad \text{for } 1 \le p < \infty$$

and $|f|_\infty$ is the essential supremum of $|f|$ on $\mathbb{R}$. Because the L^2 norm appears frequently below, we use the special notation $\|f\|$ for it. That is,

$$\|f\| = \left(\int_{-\infty}^{\infty} |f|^2 \, dx \right)^{1/2}.$$

We say that a function f defined on $\mathbb{R}$ is C^∞ if f and all its derivatives of all orders exist everywhere on $\mathbb{R}$.

We denote by $H^1_{\mathbb{C}} = H^1_{\mathbb{C}}(\mathbb{R})$ the Sobolev space of all complex-valued functions f defined on $\mathbb{R}$ such that f and its distributional derivative f' are both in L^2. The norm $\| \cdot \|_1$ on $H^1_{\mathbb{C}}$ is defined by

$$\|f\|_1 = \left(\int_{-\infty}^{\infty} \left(|f|^2 + |f'|^2 \right) \, dx \right)^{1/2}.$$

We denote the space of all real-valued functions f in $H^1_{\mathbb{C}}$ by H^1, and we define Y to be the product space

$$Y = H^1_{\mathbb{C}} \times H^1,$$

furnished with the product norm, which we denote by $\|.\|_Y$. That is,

$$\|(h,g)\|_Y^2 = \|h\|_1^2 + \|g\|_1^2.$$

We occasionally use below the operation of convolution of two functions, here denoted by the symbol $\star$ and defined by

$$f \star g(x) = \int_{-\infty}^{\infty} f(x-y) g(y) \, dy. \tag{1.16}$$

In the estimates below, the letter C will frequently be used to denote various constants whose actual values are not important for our purposes. In particular, the value of C may differ from line to line.

2. Existence of Solitary-Wave Solutions

In this section, we prove Theorem 1.1. We assume throughout the section, unless otherwise stated, that the assumptions of Theorem 1.1 hold for the constants α, τ_1, τ_2, p, q, s, and t.

Lemma 2.1. *Every minimizing sequence for $I(s,t)$ is bounded in Y. Furthermore, one has $-\infty < I(s,t) < 0$.*

Proof. First, observe that if $\{(f_n, g_n)\}$ is a minimizing sequence for $I(s,t)$, then $\|f_n\|$ and $\|g_n\|$ are bounded. From the Gagliardo-Nirenberg inequality (see, for example, Theorem 9.3 of [13]), we have that

$$|f_n|_{q+2}^{q+2} \le C\|f_{nx}\|^{q/2}\|f_n\|^{(q+4)/2}, \tag{2.1}$$

and since $\|f_n\|$ is constant, it follows that

$$|f_n|_{q+2}^{q+2} \le C\|(f_n, g_n)\|_Y^{q/2}. \tag{2.2}$$

Similarly,

$$|g_n|_{p+2}^{p+2} \le C\|g_{nx}\|^{p/2} \le C\|(f_n, g_n)\|_Y^{p/2}. \tag{2.3}$$

(Here, as throughout the paper, C denotes various constants which may depend on s and t but are independent of f_n and g_n.) Moreover, the same estimate (2.2) with q replaced by 2 shows that

$$|f_n|_4^4 \le C\|f_{nx}\| \cdot \|f_n\|^3 \le C\|f_{nx}\|,$$

so by Hölder's inequality,

$$\int_{-\infty}^{\infty} |f_n|^2|g_n|\ dx \le |f_n|_4^2 \cdot \|g_n\| \le C\|f_{nx}\|^{1/2} \le C\|(f_n, g_n)\|_Y^{1/2}. \tag{2.4}$$

Now

$$\begin{aligned}\|(f_n, g_n)\|_Y^2 &= \|f_n\|_1^2 + \|g_n\|_1^2 \\ &= E(f_n, g_n) + \int_{-\infty}^{\infty} \left(\beta_1|f_n|^{q+2} + \beta_2 g_n^{p+2} + \alpha|f_n|^2 g_n\right)\ dx + \|f_n\|^2 + \|g_n\|^2,\end{aligned}$$

and $E(f_n, g_n)$ is bounded since $\{(f_n, g_n)\}$ is a minimizing sequence. Therefore from (2.2), (2.3), and (2.4) it follows that

$$\|(f_n, g_n)\|_Y^2 \le C\left(1 + \|(f_n, g_n)\|_Y^{1/2} + \|(f_n, g_n)\|_Y^{q/2} + \|(f_n, g_n)\|_Y^{p/2}\right).$$

Since $q/2 < 2$ and $p/2 < 2$, we deduce that $\|(f_n, g_n)\|_Y$ is bounded.

Once we have shown that $\{(f_n, g_n)\}$ is bounded in Y, a finite lower bound on $E(f_n, g_n)$ also follows immediately from (2.2), (2.3), and (2.4). So $I(s,t) > -\infty$.

Finally, to see that $I(s,t) < 0$, choose $(f,g) \in Y$ such that $\|f\|^2 = s$, $\|g\|^2 = t$, and $f(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}$. For each $\theta > 0$, the functions $f_\theta(x) = \theta^{1/2} f(\theta x)$ and $g_\theta(x) = \theta^{1/2} g(\theta x)$ satisfy $\|f_\theta\|^2 = s$, $\|g_\theta\|^2 = t$, and

$$\begin{aligned} E(f_\theta, g_\theta) &= \int_{-\infty}^{\infty} \left(|f_{\theta x}|^2 + g_{\theta x}^2 - \beta_1|f_\theta|^{q+2} - \beta_2 g_\theta^{p+2} - \alpha|f_\theta|^2 g_\theta\right)\ dx \\ &\le \theta^2 \int_{-\infty}^{\infty} \left(|f_x|^2 + g_x^2\right)\ dx - \theta^{1/2}\int_{-\infty}^{\infty} \alpha|f|^2 g\ dx.\end{aligned}$$

Hence, by taking θ sufficiently small, we get $E(f_\theta, g_\theta) < 0$, proving that $I(s,t) < 0$. $\square$

Lemma 2.2. *Suppose (f_n, g_n) is a minimizing sequence for $I(s,t)$, where $t > 0$ and $s \geq 0$. (Note that we do not require $s > 0$ here.) Then there exists $\delta > 0$ such that $\|g_{nx}\| \geq \delta$ for all sufficiently large n.*

Proof. If the conclusion is not true, then by passing to a subsequence we may assume there exists a minimizing sequence for which $\lim_{n\to\infty} \|g_{nx}\| = 0$. From (2.3) it then follows that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n^{p+2}\ dx = 0.$$

Moreover, because of the elementary estimate

$$|g_n|_\infty \leq C\|g_n\|^{1/2}\|g_{nx}\|^{1/2},$$

we can write, in place of (2.4),

$$\int_{-\infty}^{\infty} |f_n|^2|g_n|\ dx \leq C\|f_n\|^2\|g_n\|^{1/2}\|g_{nx}\|^{1/2} \leq C\|g_{nx}\|^{1/2}, \tag{2.5}$$

from which it follows that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n|^2 g_n\ dx = 0.$$

Hence

$$\begin{aligned} I(s,t) &= \lim_{n\to\infty} E(f_n, g_n) \\ &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1 |f_n|^{q+2}\right)\ dx. \end{aligned} \tag{2.6}$$

Now let ψ be any non-negative function such that $\|\psi\|^2 = t$. For every $\theta > 0$, the function $\psi_\theta(x) = \theta^{1/2}\psi(\theta x)$ satisfies $\|\psi_\theta\|^2 = t$, so that $I(s,t) \leq E(f_n, \psi_\theta)$ for all n. On the other hand, if we define

$$\eta = \theta^2 \int_{-\infty}^{\infty} \psi_x^2\ dx - \beta_2\theta^{p/2}\int_{-\infty}^{\infty} \psi^{p+2}\ dx, \tag{2.7}$$

then since $p/2 < 1$, by fixing $\theta > 0$ sufficiently small we can arrange that

$$\eta < 0. \tag{2.8}$$

Then for all $n \in \mathbb{N}$,

$$\begin{aligned} I(s,t) &\leq E(f_n, \psi_\theta) \\ &= \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1|f_n|^{q+2} - \theta^{1/2}\alpha|f_n|^2\psi\right)\ dx + \eta \\ &\leq \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1|f_n|^{q+2}\right)\ dx + \eta. \end{aligned}$$

Therefore

$$I(s,t) \leq \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1|f_n|^{q+2}\right)\ dx + \eta,$$

which contradicts (2.6) and (2.8). □

Lemma 2.3. *Suppose $g(x)$ is an integrable function on $\mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} g(x)\ dx > 0. \tag{2.9}$$

Then for every $s > 0$ there exists $f \in H^1$ such that $\|f\|^2 = s$ and

$$\int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g\right)\ dx < 0.$$

Proof. Let ψ be an arbitrary smooth, non-negative function with compact support such that $\psi(0) = 1$ and $\|\psi\|^2 = s$, and for $\theta > 0$ define $\psi_\theta(x) = \theta^{1/2}\psi(\theta x)$. Then $\|\psi_\theta\|^2 = s$, and

$$\int_{-\infty}^{\infty} \left(\psi_{\theta x}^2 - \psi_\theta^2 g\right)\ dx = \theta^2 \int_{-\infty}^{\infty} \psi_x^2\ dx - \theta \int_{-\infty}^{\infty} \psi(\theta x)^2 g(x)\ dx. \tag{2.10}$$

But, by the Dominated Convergence Theorem,

$$\lim_{\theta \to 0} \int_{-\infty}^{\infty} \psi(\theta x)^2 g(x)\ dx = B,$$

where $B = \displaystyle\int_{-\infty}^{\infty} g(x)\ dx > 0$. Therefore from (2.10) it follows that

$$\int_{-\infty}^{\infty} \left(\psi_{\theta x}^2 - \psi_\theta^2 g\right)\ dx \le \theta^2 \int_{-\infty}^{\infty} \psi_x^2\ dx - \theta B/2 \tag{2.11}$$

for all θ in some neighborhood of 0. Since the quantity on the right-hand side can be made negative by taking θ sufficiently small, the desired f can be found by taking $f = \psi_\theta$ for a sufficiently small value of θ. □

Lemma 2.4. *Define $J : H^1 \to \mathbb{R}$ by*

$$J(g) = \int_{-\infty}^{\infty} \left(g_x^2 - \beta_2 g^{p+2}\right)\ dx. \tag{2.12}$$

Let $t > 0$, and let $\{g_n\}$ be any sequence of functions in H^1 such that

$$\lim_{n\to\infty} \|g_n\|^2 = t,$$

and

$$\lim_{n\to\infty} J(g_n) = \inf\ \left\{J(g) : g \in H^1 \text{ and } \|g\|^2 = t\right\}.$$

Then there exists a subsequence $\{g_{n_k}\}$ and a sequence of real numbers y_k such that $g_{n_k}(x + y_k)$ converges strongly in H^1 norm to $g_0(x)$, where

$$g_0(x) = \left(\frac{\lambda}{\beta_2}\right)^{1/p} \operatorname{sech}^{2/p}\left(\frac{\sqrt{\lambda} p x}{2}\right), \tag{2.13}$$

and $\lambda > 0$ is chosen so that $\|g_0\|^2 = t$. In particular,

$$J(g_0) = \inf\ \left\{J(g) : g \in H^1 \text{ and } \|g\|^2 = t\right\}. \tag{2.14}$$

Proof. The proof that some subsequence of g_n must converge, after suitable translations, strongly in H^1 norm is by now a standard exercise in the use of the method of concentration compactness. A proof in the case $p = 1$ appears, for example, in Theorem 2.9 of [1], or Theorem 3.13 of [2]. A similar proof, with obvious alterations, works for all $p \in [1, 4)$ because for such p the Gagliardo-Nirenberg inequality (2.3) permits one to obtain a uniform bound on $\|g_n\|_1$.

Denote the translated subsequence of $\{g_n\}$ which converges strongly by $\{g_{n_k}(x + \tilde{y}_k)\}$, and let $\psi \in H^1$ be its limit. Then ψ must satisfy

$$J(\psi) = \inf \left\{ J(g) : g \in H^1 \text{ and } \|g\|^2 = t \right\}, \tag{2.15}$$

and must also be a solution of the Euler-Lagrange equation

$$-2\psi'' - (p + 2)\beta_2 \psi^{p+1} = -2\lambda\psi \tag{2.16}$$

for some real number λ. Equation (2.16) can be explicitly integrated to show that, in order for ψ to be in H^1, λ must be positive and ψ must be a translate of the function g_0 defined in (2.13), say $\psi(x) = g_0(x + y_0)$ for some $y_0 \in \mathbb{R}$. Then (2.14) follows from (2.15). Also, defining $y_k = \tilde{y}_k - y_0$, we have that $g_{n_k}(x + y_k)$ converges to g_0 in H^1. □

Lemma 2.5. *Suppose $\beta_1 > 0$, and define $\tilde{J} : H^1_{\mathbb{C}} \to \mathbb{R}$ by*

$$\tilde{J}(f) = \int_{-\infty}^{\infty} \left(|f_x|^2 - \beta_1 |f|^{q+2} \right) \, dx. \tag{2.17}$$

Let $s > 0$, and let $\{f_n\}$ be any sequence of functions in $H^1_{\mathbb{C}}$ such that

$$\lim_{n \to \infty} \|f_n\|^2 = s,$$

and

$$\lim_{n \to \infty} \tilde{J}(f_n) = \inf \left\{ \tilde{J}(f) : f \in H^1_{\mathbb{C}} \text{ and } \|f\|^2 = s \right\}.$$

Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$, a sequence of real numbers y_k, and a real number θ such that $e^{-i\theta} f_{n_k}(x + y_k)$ converges strongly in $H^1_{\mathbb{C}}$ norm to $f_0(x)$, where

$$f_0(x) = \left(\frac{\lambda}{\beta_1} \right)^{1/q} \operatorname{sech}^{2/q} \left(\frac{\sqrt{\lambda} p x}{2} \right), \tag{2.18}$$

and $\lambda > 0$ is chosen so that $\|f_0\|^2 = s$. In particular,

$$\tilde{J}(f_0) = \inf \left\{ \tilde{J}(f) : f \in H^1_{\mathbb{C}} \text{ and } \|f\|^2 = s \right\}. \tag{2.19}$$

Proof. The comments in the first paragraph of the proof of Lemma 2.4 apply as well to $\tilde{J}$ as to J, since the proof alluded to there works here with no formal changes: the only difference is that now $\|f_n\|$ represents the modulus of a complex-valued function. Therefore we can conclude that there exists a subsequence $\{f_{n_k}\}$ and a sequence of real numbers $\tilde{y}_k$ such that $\{f_{n_k}(x+\tilde{y}_k)\}$ converges strongly in $H^1_{\mathbb{C}}$ to a (now complex-valued) function ϕ for which

$$\tilde{J}(\phi) = \inf \left\{ J(f) : f \in H^1_{\mathbb{C}} \text{ and } \|f\|^2 = t \right\}, \tag{2.20}$$

and for which the Euler-Lagrange equation

$$-2\phi'' - (q+2)\beta_1 \phi^{q+1} = -2\lambda\phi \tag{2.21}$$

holds, where here λ is again a real number.

It is proved in Theorem 8.1.6 of [7] that for every solution ϕ of (2.21), there exists a real number θ such that $\phi(x) = e^{i\theta}\tilde{\phi}(x)$ on $\mathbb{R}$, where $\tilde{\phi}(x)$ is real-valued and positive (the same argument used there is also given below in the proof of part (iv) of Theorem 1.1). The H^1 function $\tilde{\phi}$ also satisfies (2.21), and so, as in the proof of Lemma 2.4, it follows that there exists $y_0 \in \mathbb{R}$ such that $\tilde{\phi}(x) = f_0(x+y_0)$ on $\mathbb{R}$, where f_0 is as defined in (2.18). Since $\tilde{J}(\phi) = \tilde{J}(\tilde{\phi})$, then (2.19) follows from (2.20). Also, if we define $y_k = \tilde{y}_k - y_0$, then we have that $e^{-i\theta} f_{n_k}(x+y_k)$ converges in $H^1_{\mathbb{C}}$ to f_0. $\square$

Lemma 2.6. *Suppose (f_n, g_n) is a minimizing sequence for $I(s,t)$, where $s > 0$ and $t \geq 0$. If $t > 0$, or $t = 0$ and $\beta_1 > 0$, then there exists $\delta > 0$ such that $\|f_{nx}\| \geq \delta$ for all sufficiently large n. If $t = 0$ and $\beta_1 = 0$, then $I(s,t) = 0$.*

Proof. As in the proof of Lemma (2.2), we argue by contradiction. If the conclusion is not true, then by passing to a subsequence we may assume there exists a minimizing sequence for which $\lim_{n\to\infty} \|f_{nx}\| = 0$. From (2.1) and (2.4) we have that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n|^2 g_n = \lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n|^{q+2} = 0, \tag{2.22}$$

so

$$I(s,t) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(g_{nx}^2 - \beta_2 g_n^{p+2} \right) \, dx. \tag{2.23}$$

In case $t > 0$, we have from (2.14) that

$$I(s,t) \geq J(g_0), \tag{2.24}$$

where g_0 is as (2.13), and therefore g_0 is integrable with positive integral. Therefore, by Lemma 2.3 there exists $f \in H^1$ such that $\|f\|^2 = s$ and

$$\int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0 \right) \, dx < 0. \tag{2.25}$$

It follows that

$$I(s,t) \leq E(f, g_0) = \int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0 - \beta_1 |f|^{q+2} \right) \, dx + J(g_0) < J(g_0), \tag{2.26}$$

which contradicts (2.24).

In case $t = 0$ and $\beta_1 > 0$, then by (2.23), $I(s,t) = 0$. On the other hand $I(s,t) = I(s,0)$ is the infimum of

$$E(f,0) = \int_{-\infty}^{\infty} \left(|f_x|^2 - \beta_1 |f|^{q+2}\right)\, dx \tag{2.27}$$

over all $f \in H^1_{\mathbb{C}}$ satisfying $\|f\|^2 = s$. Let f be any non-negative function in H^1 such that $\|f\|^2 = s$, and define $f_\theta(x) = \theta^{1/2} f(\theta x)$. Then

$$E(f_\theta, 0) = \theta^2 \int_{-\infty}^{\infty} f_x^2\, dx - \beta_1 \theta^{q/2} \int_{-\infty}^{\infty} f^{q+2}\, dx, \tag{2.28}$$

and since $q < 4$, we can make the right-hand side negative by choosing a sufficiently small value of θ. Therefore $I(s,t) < 0$, giving a contradiction.

Finally, if $t = 0$ and $\beta_1 = 0$, then $I(s,t) = I(s,0)$ is the infimum of

$$E(f,0) = \int_{-\infty}^{\infty} |f_x|^2\, dx \tag{2.29}$$

over all f in $H^1_{\mathbb{C}}$ such that $\|f\|^2 = s$. This infimum is clearly non-negative, but on the other hand if we replace f by f_θ, as defined in the preceding paragraph, then we can make $E(f_\theta, 0)$ arbitrarily small by taking θ sufficiently small. Hence $I(s,t) = 0$. □

Lemma 2.7. *Suppose (f_n, g_n) is a minimizing sequence for $I(s,t)$, where $s > 0$ and $t > 0$. Then there exists $\delta > 0$ such that for all sufficiently large n,*

$$\int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1 |f_n|^{q+2} - \alpha |f_n|^2 g_n\right)\, dx \le -\delta.$$

Proof. If the conclusion is false, then by passing to a subsequence we may assume that there exists a minimizing sequence (f_n, g_n) for which

$$\liminf_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1 |f_n|^{q+2} - \alpha |f_n|^2 g_n\right)\, dx \ge 0, \tag{2.30}$$

and so

$$I(s,t) = \lim_{n\to\infty} E(f_n, g_n) \ge \liminf_{n\to\infty} \int_{-\infty}^{\infty} \left(g_{nx}^2 - \beta_2 g_n^{p+2}\right)\, dx. \tag{2.31}$$

Define J and g_0 as in Lemma 2.4. Then (2.31) implies that

$$I(s,t) \ge J(g_0). \tag{2.32}$$

On the other hand, by Lemma 2.3, there exists $f \in H^1$ such that $\|f\|^2 = s$ and

$$\int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0\right)\, dx < 0.$$

Therefore

$$I(s,t) \le E(f, g_0) \le \int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0\right)\, dx + J(g_0) < J(g_0), \tag{2.33}$$

which contradicts (2.32). □

Lemma 2.8. *For all $(f,g) \in Y$, one has $E(|f|,|g|) \le E(f,g)$.*

Proof. It is a standard fact from analysis that if $f \in H^1_{\mathbb{C}}$, then $|f(x)|$ is in H^1 and

$$\int_{-\infty}^{\infty} ||f|_x|^2 \, dx \le \int_{-\infty}^{\infty} |f_x|^2 \, dx. \tag{2.34}$$

(For a proof, the reader may consult Theorem 6.17 of [16].) Since β_1, β_2, and α are non-negative numbers, the Lemma follows immediately. □

The next two lemmas state that $E(f,g)$ decreases when f and g are replaced by $|f|$ and $|g|$, and when $|f|$ and $|g|$ are symmetrically rearranged. Recall that, for a non-negative function $w : \mathbb{R} \to [0,\infty)$, if $\{x : w(x) > y\}$ has finite measure $m(w,y)$ for all $y > 0$, then the symmetric decreasing rearrangement w^* of w is defined by

$$w^*(x) = \inf \left\{ y \in (0,\infty) : \frac{1}{2} m(w,y) \le x \right\} \tag{2.35}$$

(or see page 80 of [16] for a different but equivalent definition). For (f,g) in Y, both $|f|$ and $|g|$ are in H^1, and hence $|f|^*$ and $|g|^*$ are well-defined.

Lemma 2.9. *For all $(f,g) \in Y$, one has $E(|f|^*,|g|^*) \le E(f,g)$.*

Proof. This follows from classic estimates on the symmetric rearrangements of functions. A basic fact about rearrangements is that they preserve L^p norms (cf. page 81 of [16]), so that

$$\int_{-\infty}^{\infty} (|f|^*)^{q+2} \, dx = \int_{-\infty}^{\infty} |f|^{q+2} \, dx \tag{2.36}$$

and

$$\int_{-\infty}^{\infty} (|g|^*)^{p+2} \, dx = \int_{-\infty}^{\infty} |g|^{p+2} \, dx. \tag{2.37}$$

Another basic inequality about rearrangements, Theorem 3.4 of [16], implies that

$$\int_{-\infty}^{\infty} (|f|^*)^2 |g|^* \, dx \ge \int_{-\infty}^{\infty} |f|^2 |g| \, dx. \tag{2.38}$$

Finally, from Lemma 7.17 of [16] we have that

$$\int_{-\infty}^{\infty} |(|f|^*)_x|^2 \, dx \le \int_{-\infty}^{\infty} ||f|_x|^2 \, dx,$$

and similarly for $g(x)$. In light of these facts, and because α, β_1, and β_2 are all non-negative, it follows from Lemma 2.8 that $E(|f|^*,|g|^*) \le E(f,g)$.

□

We will also make crucial use of the following Lemma, due to Garrisi [11] (see also the N-dimensional version given in Byeon [6]). We include a proof here since our version of the lemma differs slightly from that stated by Garrisi.

Lemma 2.10. *Suppose u and v are non-negative, even, C^∞ functions with compact support in $\mathbb{R}$, which are non-increasing on $\{x : x \geq 0\}$. Let x_1 and x_2 be numbers such that $u(x+x_1)$ and $v(x+x_2)$ have disjoint supports, and define*

$$w(x) = u(x + x_1) + v(x + x_2).$$

Let $w^* : \mathbb{R} \to \mathbb{R}$ be the symmetric decreasing rearrangement of w. Then the distributional derivative $(w^*)'$ of w^* is in L^2, and satisfies

$$\|(w^*)'\|^2 \leq \|w'\|^2 - \frac{3}{4} \min\{\|u'\|^2, \|v'\|^2\}. \tag{2.39}$$

Proof. First consider the case when $u'(x) < 0$ for all $x \in (0, c)$ and $v'(x) < 0$ for all $x \in (0, d)$, where $[-c, c]$ is the support of u and $[-d, d]$ is the support of v. Let $a = \sup\{u(x) : x \in \mathbb{R}\}$ and $b = \sup\{v(x) : x \in \mathbb{R}\}$. By interchanging u and v if necessary, we may assume that $a \leq b$.

Define $z_u : [0, \infty) \to [0, c]$ by

$$z_u(y) = \inf\{x \in [0, \infty) : u(x) \leq y\}. \tag{2.40}$$

For $y \in (0, a)$, $z_u(y)$ is equal to the unique number $x(y) \in (0, c)$ such that $u(x(y)) = y$. The function z_u is differentiable on $(0, a)$, with derivative

$$z_u'(y) = \frac{1}{u'(x(y))} < 0,$$

and we have

$$\begin{aligned}
\|u'\|^2 &= 2 \int_0^c (u'(x))^2 \, dx \\
&= 2 \int_0^a \frac{-1}{z_u'(y)} \, dy \\
&= 2 \int_0^a \frac{1}{|z_u'(y)|} \, dy.
\end{aligned}$$

For $y \geq a$ we have $z_u(y) = 0$.

Similarly, we define $z_v : [0, \infty) \to [0, d]$ by

$$z_v(y) = \inf\{x \in [0, \infty) : v(x) \leq y\}. \tag{2.41}$$

Then

$$y_v'(v(x)) = \frac{1}{v'(x)} < 0$$

on $(0, d)$, and

$$\|v'\|^2 = 2 \int_0^b \frac{1}{|z_v'(y)|} \, dy.$$

Now, for each $y \in [0, \infty)$, define

$$z(y) = z_u(y) + z_v(y). \tag{2.42}$$

Then z is continuous on $[0, \infty)$ and differentiable, with strictly negative derivative, on $(0, a)$ and on (a, b). Therefore z is strictly decreasing on $[0, b]$, and so its restriction to $[0, b]$ has an inverse function $z^{-1} : [0, c + d] \to [0, b]$, with $z^{-1}([0,c]) = [a,b]$ and $z^{-1}([c,c+d]) = ([0,a])$. From (2.35) and the definition of w, using the fact that $u(x+x_1)$ and $v(x+x+2)$ have disjoint supports, we see that w^* is supported on $[0,c+d]$ and coincides with z^{-1} there. In particular, for all $y \in (0,a) \cup (a,b)$, we have

$$(w^*)'(z(y)) = \frac{1}{z_u'(y) + z_v'(y)}.$$

Now making use of the fact that for all positive numbers μ and ν, there holds the elementary inequality

$$\frac{2}{\mu+\nu} \leq \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{\nu}\right),$$

we have the following computation:

$$\begin{aligned}
\|(w^*)'\|^2 &= 2\int_0^{c+d} ((w^*)'(x))^2\, dx \\
&= 2\int_0^{c} ((w^*)'(x))^2\, dx + 2\int_c^{c+d} ((w^*)'(x))^2\, dx \\
&= 2\int_0^{a} \frac{1}{|z_u'(y)| + |z_v'(y)|}\, dy + 2\int_a^{b} \frac{1}{|z_v'(y)|}\, dy \\
&\leq \frac{1}{2}\int_0^{a} \left(\frac{1}{|z_u'(y)|} + \frac{1}{|z_v'(y)|}\right) dy + 2\int_a^{b} \frac{1}{|z_v'(y)|}\, dy \\
&< \frac{1}{2}\int_0^{a} \frac{1}{|z_u'(y)|}\, dy + 2\int_0^{a} \frac{1}{|z_v'(y)|}\, dy + 2\int_a^{b} \frac{1}{|z_v'(y)|}\, dy \\
&= \frac{1}{2}\int_0^{a} \frac{1}{|z_u'(y)|}\, dy + 2\int_0^{b} \frac{1}{|z_v'(y)|}\, dy \\
&= \frac{1}{2}\int_0^{c} (u'(x))^2\, dx + 2\int_0^{d} (v'(x))^2\, dx \\
&= 2\int_0^{c} (u'(x))^2\, dx + 2\int_0^{d} (v'(x))^2\, dx - \frac{3}{2}\int_0^{c} (u'(x))^2\, dx \\
&= \frac{1}{2}\|u'\|^2 + \frac{1}{2}\|v'\|^2 - \frac{3}{4}\|u'\|^2 \\
&= \frac{1}{2}\|w'\|^2 - \frac{3}{4}\|u'\|^2 \\
&\leq \frac{1}{2}\|w'\|^2 - \frac{3}{4}\min\{\|u'\|^2, \|v'\|^2\}.
\end{aligned}$$

Thus (2.39) is proved in the special case when $u' < 0$ on $(0,c)$ and $v' < 0$ on $(0,d)$.

Now we consider the general case, which we can reduce to the case treated above as follows.

Let $\phi_1(x)$ be a smooth, even function such that $\phi_1(x) > 0$ for $x \in (0,c)$ and $\phi_1(x) = 0$ for $x \geq c$, and such that $\phi_1(x)$ is strictly decreasing on $(0,c)$. Let $\phi_2(x)$ be a similar function with support on $(0,d)$. For each $\epsilon > 0$, define $u_\epsilon(x) = u(x) + \epsilon\phi_1(x)$ and $v_\epsilon(x) = v(x) + \epsilon\phi_2(x)$, and let $w_\epsilon(x) = u_\epsilon(x)+v_\epsilon(x-T)$. Since $u' \leq 0$ and $\phi_1' < 0$ on $(0,c)$, then $u_\epsilon' = u'+\epsilon\phi_1' < 0$ on $(0,c)$, so u_ϵ is strictly decreasing on $(0,c)$. Similarly, v_ϵ is strictly decreasing on $(0,d)$. So, by what has been proved above,

$$\|(w_\epsilon^*)'\|^2 \leq \|w_\epsilon'\|^2 - \frac{3}{4}\min\{\|u_\epsilon'\|^2, \|v_\epsilon'\|^2\}. \tag{2.43}$$

Now take limits on both sides of (2.43) as ϵ goes to zero. By the dominated convergence theorem, the right hand side approaches

$$\|w'\|^2 - \frac{3}{4}\min\{\|u'\|^2, \|u'\|^2\}.$$

Also, since w_ϵ converges in H^1 norm on $\mathbb{R}$ to w, then by a theorem of Coron [10], w_ϵ^* converges in H^1 norm to w^*. Therefore the left-hand side of (2.39) converges to $\|(w^*)'\|^2$, and (2.39) is proved. □

Lemma 2.11. *The functionals E, G, and H are continuous from Y to $\mathbb{R}$.*

Proof. This follows easily (for all $p \geq 0$ and $q \geq 0$) from the Sobolev embedding theorem, in particular using the fact that the inclusion of H^1 in L^∞ is continuous. □

Lemma 2.12. *Let $s_1, s_2, t_1, t_2 \geq 0$ be given, and suppose that $s_1 + s_2 > 0$, $t_1 + t_2 > 0$, $s_1 + t_1 > 0$, and $s_2 + t_2 > 0$. Then*

$$I(s_1 + s_2, t_1 + t_2) < I(s_1, t_1) + I(s_2, t_2). \tag{2.44}$$

Proof. We claim first that, for $i = 1, 2$, we can choose minimizing sequences $(f_n^{(i)}, g_n^{(i)})$ for $I(s_i, t_i)$ such that for all $n \in \mathbb{N}$, $f_n^{(i)}$ and $g_n^{(i)}$

(i) are real-valued and non-negative on $\mathbb{R}$;
(ii) belong to H^1 and have compact support;
(iii) are even functions;
(iv) are non-increasing functions of x for $x \geq 0$;
(v) are C^∞ functions; and
(vi) satisfy $\|f_n^{(i)}\| = s_i$ and $\|g_n^{(i)}\| = t_i$.

To prove this, we can take $i = 1$, since the proof for $i = 2$ is identical. Also we may assume that $s_1 > 0$ and $t_1 > 0$, since otherwise we can simply take $f_n^{(1)}$ or $g_n^{(1)}$ to be identically zero on $\mathbb{R}$.

Start with an arbitrary minimizing sequence $(w_n^{(1)}, z_n^{(1)})$ for $I(s_1, t_1)$. Since functions with compact support are dense in H^1, and $E : Y \to \mathbb{R}$ is continuous, we can approximate $(w_n^{(1)}, z_n^{(1)})$ by functions $(w_n^{(2)}, z_n^{(2)})$ which have compact support and which still form a minimizing sequence for $I(s_1, t_1)$. Then from Lemma 2.9 it follows that the sequence defined by

$$(w_n^{(3)}, z_n^{(3)}) = (|w_n^{(2)}|^*, |z_n^{(2)}|^*)$$

is still a minimizing sequence for $I(s_1, t_1)$, and for each n, $w_n^{(3)}$ and $z_n^{(3)}$have the properties (i) through (iv) listed above.

Next, observe that if f and ψ are any two functions with properties (i) through (iv), then their convolution $f \star \psi$, defined as in (1.16), also satisfies properties (i) through (iv). Moreover, as is well known, if we define $\psi_\epsilon = (1/\epsilon)\psi(x/\epsilon)$ for $\epsilon > 0$, and choose ψ such that $\int_{-\infty}^{\infty} \psi(x)\ dx = 1$, then convolution with ψ_ϵ is an "approximation to the identity": that is, the functions $f \star \psi_\epsilon$ converge strongly to f in H^1 as $\epsilon \to 0$. Finally, if ψ is C^∞ then $f \star \psi_\epsilon$ will be C^∞ also. Therefore by choosing $\psi(x)$ to be any non-negative, C^∞, even function with compact support, which is decreasing for $x \geq 0$, and satisfies $\int_{-\infty}^{\infty} \psi(x)\ dx = 1$, and defining

$$(w_n^{(4)}, z_n^{(4)}) = (w_n^{(3)} \star \psi_{\epsilon_n}, z_n^{(3)} \star \psi_{\epsilon_n}),$$

with ϵ_n chosen appropriately small for n large, we obtain a minimizing sequence $(w_n^{(4)}, z_n^{(4)})$ for $I(s_1, t_1)$ that satisfies not only the properties (i) through (iv) above, but also property (v).

Finally, we obtain the desired minimizing sequence satisfying properties (i) through (vi) by setting

$$f_n^{(1)} = \frac{(s_i)^{1/2} w_n^{(4)}}{\|w_n^{(4)}\|} \quad \text{and} \quad g_n^{(1)} = \frac{(t_i)^{1/2} z_n^{(4)}}{\|g_n^{(i)}\|},$$

respectively, which is possible since for n sufficiently large we have $\|w_n^{(4)}\| > 0$ and $\|z_n^{(4)}\| > 0$.

Next, choose for each n a number x_n such that $f_n^{(1)}(x)$ and $\tilde{f}_n^{(2)}(x) = f_n^{(2)}(x + x_n)$ have disjoint support, and $g_n^{(1)}(x)$ and $\tilde{g}_n^{(2)}(x) = g_n^{(2)}(x + x_n)$ have disjoint support. Define

$$\begin{aligned} f_n &= \left(f_n^{(1)} + \tilde{f}_n^{(2)}\right)^*, \\ g_n &= \left(g_n^{(1)} + \tilde{g}_n^{(2)}\right)^*. \end{aligned}$$

Then $\|f_n\|^2 = s_1 + s_2$ and $\|g_n\|^2 = t_1 + t_2$, so

$$I(s_1 + s_2, t_1 + t_2) \leq E(f_n, g_n). \tag{2.45}$$

On the other hand, from Lemma 2.10 we have that

$$\begin{aligned} \int_{-\infty}^{\infty} \left(f_{nx}^2 + g_{nx}^2\right)\ dx &\leq \int_{-\infty}^{\infty} \left((f_n^{(1)} + \tilde{f}_n^{(2)})_x^2 + (g_n^{(1)} + \tilde{g}_n^{(2)})_x^2\right)\ dx - K_n \\ &= \int_{-\infty}^{\infty} \left((f_{nx}^{(1)})^2 + (\tilde{f}_{nx}^{(2)})^2 + (g_{nx}^{(1)})^2 + (\tilde{g}_{nx}^{(2)})^2\right)\ dx - K_n, \end{aligned} \tag{2.46}$$

where

$$K_n = \frac{3}{4}\left(\min\left\{\|f_{nx}^{(1)}\|^2, \|f_{nx}^{(2)}\|^2\right\} + \min\left\{\|g_{nx}^{(1)}\|^2, \|g_{nx}^{(2)}\|^2\right\}\right). \tag{2.47}$$

Furthermore, from the properties (2.37), (2.36), and (2.38) of rearrangements, we have that

$$
\begin{aligned}
\int_{-\infty}^{\infty} |f_n|^{q+2}\,dx &= \int_{-\infty}^{\infty} |f_n^{(1)}|^{q+2}\,dx + \int_{-\infty}^{\infty} |f_n^{(2)}|^{q+2}\,dx \\
\int_{-\infty}^{\infty} g_n^{p+2}\,dx &= \int_{-\infty}^{\infty} (g_n^{(1)})^{p+2}\,dx + \int_{-\infty}^{\infty} (g_n^{(2)})^{q+2}\,dx \\
\int_{-\infty}^{\infty} |f_n|^2 g_n\,dx &\geq \int_{-\infty}^{\infty} |f_n^{(1)}|^2 g_n^{(1)}\,dx + \int_{-\infty}^{\infty} |f_n^{(2)}|^2 g_n^{(2)}\,dx,
\end{aligned}
\tag{2.48}
$$

and therefore, combining with (2.45) and (2.46), we have that for every n,

$$
I(s_1+t_1, s_2+t_2) \leq E(f_n, g_n) \leq E(f_n^{(1)}, g_n^{(1)}) + E(f_n^{(2)}, g_n^{(2)}) - K_n. \tag{2.49}
$$

It follows by taking the limit superior on the right-hand side that

$$
I(s_1+t_1, s_2+t_2) \leq I(s_1, t_1) + I(s_2, t_2) - \liminf_{n\to\infty} K_n. \tag{2.50}
$$

Since $t_1 + t_2 > 0$, then either t_1 and t_2 are both positive, or one of t_1 and t_2 is zero and the other is positive. In the latter case, we may assume that $t_1 = 0$ and $t_2 > 0$, since otherwise we can simply switch t_1 and t_2. Then we will argue separately according as to whether s_2 is positive or zero. To prove the theorem, then, it suffices to consider the following three cases: (i) $t_1 > 0$ and $t_2 > 0$; (ii) $t_1 = 0$, $t_2 > 0$, and $s_2 > 0$; and (iii) $t_1 = 0$, $t_2 > 0$, and $s_2 = 0$.

In case (i), when $t_1 > 0$ and $t_2 > 0$, it follows from Lemma 2.2 that there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that for all sufficiently large n, $\|(g_n^{(1)})_x\| \geq \delta_1$ and $\|(g_n^{(2)})_x\| \geq \delta_2$. (Note that by Lemma 2.6, this is still true even when $s_1 = 0$ or $s_2 = 0$.) So, letting $\delta = \min(\delta_1, \delta_2) > 0$, (2.47) gives $K_n \geq 3\delta/4$ for all sufficiently large n. From (2.50) we then have that

$$
I(s_1+t_1, s_2+t_2) \leq I(s_1, t_1) + I(s_2, t_2) - 3\delta/4 < I(s_1, t_1) + I(s_2, t_2), \tag{2.51}
$$

as desired.

In case (ii), we have $t_1 = 0$, $t_2 > 0$, $s_2 > 0$, and, since $s_1 + t_1 > 0$ by assumption, $s_1 > 0$ also. By Lemma 2.6 there exists $\delta_1 > 0$ such that for all sufficiently large n, $\|(f_n^{(1)})_x\| \geq \delta_1$.

If, in case (ii), $\beta_1 > 0$, then by Lemma 2.6 there also exists $\delta_2 > 0$ such that for all sufficiently large n, $\|f_{nx}^{(2)}\| \geq \delta_2$. Letting $\delta = \min(\delta_1, \delta_2) > 0$, we get $K_n \geq 3\delta/4$ for large n, and (2.51) follows from (2.50) as in case (i).

On the other hand, if in case (ii) we have $\beta_1 = 0$, then by Lemma 2.6 we have $I(s_1, t_1) = I(s_1, 0) = 0$, and $I(s_1 + s_2, t_1 + t_2) = I(s_1 + s_2, t_2)$ is the infimum of

$$
E(f, g) = \int_{-\infty}^{\infty} \left(|f_x|^2 + g_x^2 - \beta_2 g^{p+2} - \alpha |f|^2 g\right)\,dx \tag{2.52}
$$

over all $f \in H^1_{\mathbb{C}}$ and $g \in H^1$ such that $\|f\|^2 = s_1 + s_2$ and $\|g\|^2 = t_2$. By Lemma 2.7, there exists $\delta > 0$ such that for all sufficiently large n,

$$\int_{-\infty}^{\infty} \left(|f_{nx}^{(2)}|^2 - \alpha |f_n^{(2)}|^2 g_n^{(2)} \right) \, dx \le -\delta.$$

Let

$$f_n = \sqrt{\frac{s_1 + s_2}{s_2}} f_n^{(2)}; \tag{2.53}$$

then $\|f_n\|^2 = s_1 + s_2$ and from (2.52) we see that, for all sufficiently large n,

$$\begin{aligned} I(s_1 + s_2, t_2) &\le E(f_n, g_n^{(2)}) = E(f_n^{(2)}, g_n^{(2)}) + \frac{s_1}{s_2} \int_{-\infty}^{\infty} \left(|f_{nx}^{(2)}|^2 - \alpha |f_n^{(2)}|^2 g_n^{(2)} \right) \, dx \\ &\le E(f_n^{(2)}, g_n^{(2)}) - \frac{s_1 \delta}{s_2}. \end{aligned} \tag{2.54}$$

This implies, after taking the limit as $n \to \infty$, that

$$I(s_1 + s_2, t_2) \le I(s_2, t_2) - \frac{s_1 \delta}{s_2} < I(s_2, t_2) = I(s_1, t_1) + I(s_2, t_2), \tag{2.55}$$

as desired. Thus the proof is complete in case (ii).

In case (iii), we have $s_1 > 0$ and $t_2 > 0$, and we have to prove

$$I(s_1, t_2) < I(s_1, 0) + I(0, t_2). \tag{2.56}$$

Let g_0 be as defined in Lemma 2.4 with $t = t_2$, so that $I(0, t_2) = J(g_0)$.

If $\beta_1 > 0$, we have from Lemma 2.5 that $I(s_1, 0) = \tilde{J}(f_0)$, where f_0 is as defined in (2.18) with $s = s_1$. Clearly,

$$\int_{-\infty}^{\infty} |f_0|^2 g_0 \, dx > 0,$$

and so

$$\begin{aligned} I(s_1, t_2) \le E(f_0, g_0) &= \tilde{J}(f_0) + J(g_0) + \int_{-\infty}^{\infty} |f_0|^2 g_0 \, dx \\ &< \tilde{J}(f_0) + J(g_0) = I(s_1, 0) + I(0, t_2), \end{aligned} \tag{2.57}$$

as desired.

On the other hand, if $\beta_1 = 0$, then $I(s_1, 0) = 0$ by Lemma 2.6. By Lemma 2.3, there exists $f \in H^1$ such that $\|f\|^2 = s_1$ and

$$\int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0 \right) \, dx < 0, \tag{2.58}$$

and hence

$$I(s_1, t_1) \le E(f, g_0) = \int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0 \right) \, dx + J(g_0) < J(g_0), \tag{2.59}$$

which proves (2.56). The proof of Lemma 2.12 is now complete in all cases. □

We now turn to the proof of Theorem 1.1, which, once the subadditivity lemma 2.12 has been established, proceeds by largely the same argument as in [2].

The first step is to establish the relative compactness, up to translations, of minimizing sequences for $I(s,t)$. Let $\{(f_n, g_n)\}$ be a given minimizing sequence, and define an associated sequence of functions ρ_n by

$$\rho_n = |f_n|^2 + g_n^2.$$

We then have

$$\int_{-\infty}^{\infty} \rho_n(x)\ dx = s + t$$

for all n. The sequence of functions $M_n : [0,\infty) \to [0, s+t]$ defined by

$$M_n(r) = \sup_{y \in \mathbb{R}} \int_{y-r}^{y+r} \rho_n(x)\ dx.$$

is a uniformly bounded sequence of nondecreasing functions on $[0,\infty)$, and therefore (by Helly's selection theorem, for example) has a subsequence, which we will still denote by M_n, that converges pointwise to a nondecreasing function M on $[0,\infty)$. Then

$$\gamma = \lim_{r \to \infty} M(r) \tag{2.60}$$

exists and satisfies $0 \le \gamma \le s + t$.

We claim now that $\gamma > 0$. To prove this, we require the following lemma.

Lemma 2.13. *Suppose w_n is a sequence of functions which is bounded in H^1 and which satisfies, for some $R > 0$,*

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} w_n^2\ dx = 0. \tag{2.61}$$

Then for every $r > 2$,

$$\lim_{n \to \infty} |w_n|_r = 0.$$

Proof. This is a special case of Lemma I.1 of part 2 of [17], but for the sake of completeness we give a proof here. Let

$$\epsilon_n = \sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} w_n^2\ dx, \tag{2.62}$$

so that $\lim_{n \to \infty} \epsilon_n = 0$. For every $y \in \mathbb{R}$, we have by standard Sobolev inequalities (see Theorem 10.1 of [13]) that

$$\int_{y-R}^{y+R} |w_n|^r\ dx \le C \left(\int_{y-R}^{y+R} |w_n|^2\ dx \right)^s \left(\int_{y-R}^{y+R} \left(w_n^2 + w_{nx}^2 \right)\ dx \right)^{1+s},$$

where $s = (r-2)/4$. It then follows from (2.61) that

$$
\begin{aligned}
\int_{y-R}^{y+R} |w_n|^r \, dx &\leq C\epsilon_n^s \left(\int_{y-R}^{y+R} \left(w_n^2 + w_{nx}^2\right) \, dx \right) \|w_n\|_1^s \\
&\leq C\epsilon^s \int_{y-R}^{y+R} \left(w_n^2 + w_{nx}^2\right) \, dx.
\end{aligned} \tag{2.63}
$$

Now if we cover $\mathbb{R}$ by intervals of length R in such a way that each point of $\mathbb{R}$ is contained in at most two of the intervals, then by summing (2.63) over all the intervals in the cover, we obtain that

$$
|w_n|_r \leq 3C\epsilon_n^s \|w_n\|_1^2 \leq C\epsilon_n^s,
$$

from which the desired result follows. □

Next we prove that

$$
\gamma \neq 0. \tag{2.64}
$$

Indeed, suppose for the sake of contradiction that $\gamma = 0$. Then (2.61) holds both for $w_n = |f_n|$ and for $w_n = g_n$. Since both $\{|f_n|\}$ and $\{g_n\}$ are bounded sequences in H^1 by Lemma 2.1, then Lemma (2.13) implies that for every $r > 2$, f_n and g_n converge to 0 in L^r norm. Since

$$
\left| \int_{-\infty}^{\infty} |f_n|^2 g_n \, dx \right| \leq |f_n|_4^{1/2} \|g_n\|
$$

and $\|g_n\|$ is bounded, it follows also that

$$
\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n|^2 g_n \, dx = 0.
$$

Hence

$$
I(s,t) = \lim_{n\to\infty} E(f_n, g_n) \geq \liminf_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}|^2 + g_{nx}^2\right) dx \geq 0, \tag{2.65}
$$

contradicting Lemma 2.1. This proves (2.64).

Lemma 2.14. *Suppose γ is defined as in* (2.60)*. Then there exist numbers $s_1 \in [0,s]$ and $t_1 \in [0,t]$ such that*

$$
\gamma = s_1 + t_1 \tag{2.66}
$$

and

$$
I(s_1,t_1) + I(s-s_1, t-t_1) \leq I(s,t). \tag{2.67}
$$

Proof. Since the proof is almost the same as the proof of Lemma 3.10 of [2], with only slight modifications, we just give an outline here, and refer to [2] for the details. Let ρ and σ be smooth functions on $\mathbb{R}$ such that $\rho^2 + \sigma^2 = 1$ on $\mathbb{R}$, and ρ is identically 1 on $[-1,1]$ and is supported in $[-2,2]$; and define $\rho_\omega(x) = \rho(x/\omega)$ and $\sigma_\omega(x) = \sigma(x/\omega)$ for $\omega > 0$. From the definition of γ it follows that for given $\epsilon > 0$, there exist $\omega > 0$ and a sequence y_n such that, after passing to a subsequence, the functions $(f_n^{(1)}(x), g_n^{(1)}(x)) =$

$\rho_\omega(x-y_n)(f_n(x),g_n(x))$ and $(f_n^{(2)}(x),g_n^{(2)}(x)) = \sigma_\omega(x-y_n)(f_n(x),g_n(x))$ satisfy $\|f_n^{(1)}\|^2 \to s_1$, $\|g_n^{(1)}\|^2 \to t_1$, $\|f_n^{(2)}\|^2 \to s-s_1$, and $\|g_n^{(2)}\|^2 \to t-t_1$ as $n \to \infty$, where $|(s_1+t_1)-\alpha| < \epsilon$, and

$$E(f_n^{(1)},g_n^{(1)}) + E(f_n^{(2)},g_n^{(2)}) \le E(f_n,g_n) + C\epsilon \tag{2.68}$$

for all n. To prove (2.68), one writes

$$\begin{aligned} E(f_n^{(1)},g_n^{(1)}) &= \int_{-\infty}^{\infty} \rho_\omega^2 \left(|f_{nx}|^2 + g_{nx}^2 - \beta_1|f_n|^{q+2} - \beta_2 g_n^{p+2} - \alpha|f_n|^2 g_n\right)\, dx \\ &\quad + \int_{-\infty}^{\infty} \left((\rho_\omega')^2(|f_n|^2+g_n^2) + 2\rho_\omega\rho_\omega'(\text{Re } f_n(\overline{f_n})_x + g_n g_{nx})\right)\, dx \\ &\quad + \int_{-\infty}^{\infty} (\rho_\omega^2 - \rho_\omega^{q+2})\beta_1|f_n|^{q+2}\, dx + \int_{-\infty}^{\infty} (\rho_\omega^2 - \rho_\omega^{p+2})\beta_2|g_n|^{p+2}\, dx \\ &\quad + \int_{-\infty}^{\infty} (\rho_\omega^2 - \rho_\omega^3)\alpha|f_n|^2 g_n\, dx, \end{aligned}$$

and observes that the last two integrals on the right hand side can be made arbitrarily uniformly small by taking ω sufficiently large. A similar estimate obtains for $E(f_n^{(2)},g_n^{(2)})$, and (2.68) follows by adding the two estimates and using $\rho_\omega^2 + \sigma_\omega^2 = 1$.

Now we show that the limit inferior as $n \to \infty$ of the left-hand side of (2.68) is greater than or equal to $I(s_1,t_1) + I(s-s_1,t-t_1)$. If s_1, t_1, $s-s_1$, and $t-t_1$ are all positive, this follows by rescaling $f_n^{(i)}$ and $g_n^{(i)}$ for $i=1,2$ so that $\|f_n^{(1)}\|^2 = s_1$, $\|g_n^{(1)}\|^2 = t_1$, $\|f_n^{(2)}\|^2 = s-s_1$, and $\|g_n^{(2)}\|^2 = t-t_1$, since the scaling factors tend to 1 as $n \to \infty$. On the other hand, if $s_1 = 0$ and $t_1 > 0$ then as in (2.65) we have

$$\begin{aligned} \lim_{n\to\infty} E(f_n^{(1)},g_n^{(1)}) &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}^{(1)}|^2 + (g_{nx}^{(1)})^2 - \beta_2 (g_n^{(1)})^{q+2}\right)\, dx \\ &\ge \liminf_{n\to\infty} \int_{-\infty}^{\infty} ((g_{nx}^{(1)})^2 - \beta_2 (g_n^{(1)})^{q+2})\, dx \ge I(0,t_1), \end{aligned}$$

and similar estimates hold if t_1, $s-s_1$, or $t-t_1$ are zero.

Taking then the limit inferior of the left-hand side and the limit of the right-hand side of (2.68) as $n \to \infty$, we obtain

$$I(s_1,t_1) + I(s-s_1,t-t_1) \le I(s,t) + C\epsilon,$$

which proves (2.67), as ϵ is arbitrary. □

We claim now that

$$\gamma = s+t. \tag{2.69}$$

Suppose to the contrary that $\gamma < s+t$. Let s_1 and t_1 be as defined in Lemma 2.14, and let $s_2 = s-s_1$ and $t_2 = t-t_1$. Then $s_2+t_2 = (s+t)-\gamma > 0$, and also (2.64) and (2.66) imply that $s_1+t_1 > 0$. Moreover, $s_1+s_2 = s > 0$ and $t_1+t_2 = t > 0$. Therefore Lemma 2.12 implies that that (2.44) holds. But this contradicts (2.67). Thus (2.69) is proved.

Once (2.69) has been established, assertion (ii) of Theorem 1.1, concerning the relative compactness of minimizing sequences up to translation, follows from general principles. We again only outline the proof here and refer the reader to [2] for more details. First, (2.69) immediately implies that for some sequence y_n of real numbers and some fixed subsequence of (f_n, g_n), denoted again by (f_n, g_n), and for every $k \in \mathbb{N}$, there exists $\omega_k \in \mathbb{R}$ such that

$$\int_{-\omega_k}^{\omega_k} \left(|f_n(x+y_n)|^2 + g_n(x+y_n)^2\right)\, dx \geq s + t - \frac{1}{k}. \tag{2.70}$$

for all $n \in \mathbb{N}$. (In other words, the measures

$$\mu_n = (|f_n(x+y_n)|^2 + g_n(x+y_n)^2)\, dx$$

form a "tight" family on $\mathbb{R}$, in the sense that for every $\epsilon > 0$, there exists a fixed compact set K such that $\mu_n(\mathbb{R}\backslash K) < \epsilon$ for all $n \in \mathbb{N}$.)

From (2.70) and the compactness of the embedding of H^1 into L^2 on finite domains, it follows that some further subsequence of $(f_n(x+y_n), g(x+y_n))$ converges, strongly in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and weakly in Y, to a limit (ϕ, ψ). Estimates such as (2.1) and (2.4), together with the weak lower semicontinuity of the Hilbert space norm in Y, imply that

$$\lim_{n\to\infty} E(f_n, g_n) \geq E(\phi, \psi); \tag{2.71}$$

but since (f_n, g_n) is a minimizing sequence, this in turn implies that

$$\lim_{n\to\infty} E(f_n, g_n) = E(\phi, \psi). \tag{2.72}$$

Therefore one has

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}|^2 + g_{nx}^2\right)\, dx = \int_{-\infty}^{\infty} \left(|\phi_x|^2 + \psi_x^2\right)\, dx,$$

so $(f_n(x+y_n), g_n(x+y_n))$ converges strongly to (ϕ, ψ) in the norm of Y.

Since (ϕ, ψ) is in the minimizing set $\mathcal{S}_{s,t}$ for $I(s,t)$, and so minimizes $E(u,v)$ subject to $H(u)$ and $H(v)$ being held constant, the Lagrange multiplier principle (see, for example, Theorem 7.7.2 of [18]) asserts that there exist real numbers σ and c such that

$$\delta E(\phi, \psi) = \sigma \delta H(\phi) + c \delta H(\psi), \tag{2.73}$$

where δ denotes the Fréchet derivative. Computing the Fréchet derivatives we see that this means that equations (1.8) hold, at least in the sense of distributions. But since the right-hand sides of the equations in (1.8) are continuous functions of the unknowns, distributional solutions are also classical solutions (cf. Lemma 1.3 of [22]). This then proves assertion (iii) of Theorem 1.1.

It remains to prove the assertions in part (iv) of Theorem 1.1.

Multiplying the first equation in (1.8) by $\overline{\phi}$ and integrating over $\mathbb{R}$, we have after an integration by parts that

$$\int_{-\infty}^{\infty} \left(|\phi'|^2 - \tau_1|\phi|^{q+2} - \alpha|\phi|^2\psi\right)\, dx = -\sigma \int_{-\infty}^{\infty} |\phi|^2\, dx = -\sigma s. \tag{2.74}$$

In particular, it follows from (2.74) that σ is real. Similarly, multiplying the second equation in (1.8) by ψ and integrating over $\mathbb{R}$ yields

$$\int_{-\infty}^{\infty} \left(|\psi'|^2 - \frac{\tau_2}{p+1}\psi^{p+2} - \frac{\alpha}{2}|\phi|^2\psi\right)\, dx = -c \int_{-\infty}^{\infty} |\psi|^2\, dx = -ct. \tag{2.75}$$

From Lemma 2.7, applied to the constant sequence $(f_n, g_n) = (\phi, \psi)$, we have that

$$\int_{-\infty}^{\infty} \left(|\phi'|^2 - \tau_1|\phi|^{q+2} - \alpha|\phi|^2\psi\right)\, dx < 0, \tag{2.76}$$

and since $\tau_1 = \beta_1(q+2)/2 > \beta_1$, it follows that the integral on the left-hand side of (2.74) is negative, and so we must have $\sigma > 0$. Therefore, a calculation with the Fourier transform shows that the operator $-\partial_x^2 + \sigma$ appearing in the first equation of (1.8) is invertible on $H^1_{\mathbb{C}}$, with inverse given by convolution with the function

$$K_\sigma(x) = \frac{1}{2\sqrt{\sigma}} e^{-\sqrt{\sigma}|x|}.$$

The first equation of (1.8) can then be rewritten in the form

$$\phi = K_\sigma \star \left(\tau_1|\phi|^q\phi + \alpha\phi\psi\right), \tag{2.77}$$

where $\star$ denotes convolution as in (1.16).

Now we observe that it follows from the first equation in (1.8) that there exist $\theta \in \mathbb{R}$ and a real-valued function $\tilde{\phi}(x)$ such that $\phi(x) = e^{i\theta}\tilde{\phi}(x)$ on $\mathbb{R}$. This is proved for the case $\tau_1 = 0$ in part (iii) of Theorem 2.1 of [2], and it is easy to check that the same proof works as well when $\tau_1 \neq 0$.

Note next that $(\tilde{\phi}, |\psi|)$ and $(|\tilde{\phi}|, |\psi|)$ are also in $\mathcal{S}_{s,t}$, as follows from Lemma (2.8). Therefore, if we let $w = |\tilde{\phi}|$, then $\tilde{\phi}$ and w satisfy the Lagrange multiplier equations

$$\begin{aligned} -\tilde{\phi}'' + \sigma\tilde{\phi} &= \tau_1 w^q \tilde{\phi} + \alpha\tilde{\phi}|\psi| \\ -w'' + \sigma w &= \tau_1 w^q w + \alpha w|\psi|. \end{aligned} \tag{2.78}$$

(That the Lagrange multiplier σ is the same in both equations follows from the fact that σ is determined by the equation (2.74), and this equation is unchanged when ϕ is replaced by w.) Multiplying the first equation by w and the second equation by $\tilde{\phi}$, and subtracting the two equations, we find that the $w\tilde{\phi}'' - \tilde{\phi}w'' = 0$. Therefore the Wronskian $w\tilde{\phi}' - \tilde{\phi}w'$ of w and $\tilde{\phi}$ is constant, and since w and $\tilde{\phi}$ are both in H^1, this constant must be zero. So w and $\tilde{\phi}$ are constant multiples of each other, and hence $\tilde{\phi}$, like w, must be of one sign on $\mathbb{R}$. By replacing θ by $\theta + i\pi$ if necessary, we can assume that $\tilde{\phi} \geq 0$ on $\mathbb{R}$.

We claim that

$$\int_{-\infty}^{\infty} |\phi|^2|\psi|\ dx = \int_{-\infty}^{\infty} |\phi|^2\psi\ dx. \tag{2.79}$$

To prove this, note that since $E(|\phi|,|\psi|) = E(|\phi|,\psi) = I(s,t)$, we have

$$\alpha \int_{-\infty}^{\infty} |\phi|^2(|\psi| - \psi)\ dx = \int_{-\infty}^{\infty} \left((|\psi_x|^2 - \psi_x^2) - \beta_2(|\psi|^{p+2} - \psi^{p+2})\right)\ dx. \tag{2.80}$$

Using (2.34), we see that the right-hand side of this equation is less than or equal to zero, so we must have

$$\alpha \int_{-\infty}^{\infty} |\phi|^2(|\psi| - \psi)\ dx \le 0 \tag{2.81}$$

also. But since the integrand is non-negative, this proves (2.79).

From (2.79) it follows that $\psi(x) \ge 0$ at every point x in $\mathbb{R}$ for which $\tilde{\phi}(x) \ne 0$. Now (2.77) implies that

$$\tilde{\phi} = K_\sigma \star \left(\tau_1|\tilde{\phi}|^q\tilde{\phi} + \alpha\tilde{\phi}\psi\right). \tag{2.82}$$

Since the convolution of K_σ with a function that is everywhere non-negative and not identically zero must produce an everywhere positive function, it follows that $\tilde{\phi}(x) > 0$ for all $x \in \mathbb{R}$. But this in turn implies that $\psi(x) \ge 0$ for all $x \in \mathbb{R}$.

Now suppose, for the sake of contradiction, that $\psi(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Then from the preceding paragraph it follows that x_0 is a point where ψ takes its minimum value over $\mathbb{R}$, and therefore we must have $\psi'(x_0) = 0$. But then standard uniqueness theory for ordinary differential equations, applied to the second equation in (1.8) viewed as an inhomogeneous equation for ψ, yields that ψ must be identically zero on its entire interval of existence about x_0, which in this case is $\mathbb{R}$. But this contradicts the fact that $\|\psi\|^2 = t > 0$. Therefore ψ must be everywhere positive on $\mathbb{R}$.

Finally, since ψ and $|\phi|$ are everywhere positive on $\mathbb{R}$, and the right-hand sides of the equations in (1.8) are infinitely differentiable functions of ϕ and ψ on the domain $\{(\phi,\psi) \in \mathbb{C} \times \mathbb{R} : |\phi| > 0 \text{ and } \psi > 0\}$, it follows from the standard theory of ordinary differential equations that any solution of (1.8) must be infinitely differentiable on its interval of existence.

This completes the proof of Theorem 1.1.

3. Stability of Solitary-Wave Solutions

In this section we prove the stability result given in Theorem 1.2 by considering the variational problem $W(s,t)$ defined in (1.11). In particular, we show that arbitrary minimizing sequences for $W(s,t)$ converge, up to subsequences and translations, to elements of the minimizing set $\mathcal{F}_{s,t}$ defined in (1.14). To do this, we relate the problem in (1.11) to that in (1.9), following the method used in [2].

Lemma 3.1. *Suppose $1 \le q < 4$ and $1 \le p < 4/3$, and let $s > 0$ and $t \in \mathbb{R}$. If $\{(h_n, g_n)\}$ is a minimizing sequence for $W(s,t)$, then $\{(h_n, g_n)\}$ is bounded in Y.*

Proof. Since $\|h_n\|^2 = H(h_n)$ is bounded, then

$$\begin{aligned} \|g_n\|^2 = \left| G(h_n, g_n) - \operatorname{Im} \int_{-\infty}^{\infty} h_n (\overline{h_n})_x \; dx \right| &\le C \left(1 + \|h_n\| \cdot \|h_{nx}\|\right) \\ &\le C \left(1 + \|(h_n, g_n)\|_Y\right), \end{aligned} \tag{3.1}$$

where C is independent of n. Therefore

$$\begin{aligned} \|(h_n, g_n)\|_Y^2 &= E(h_n, g_n) + \int_{-\infty}^{\infty} \left(\beta_1 |h_n|^{q+2} + \beta_2 g_n^{p+2} + \alpha |h_n|^2 g_n\right) \; dx + \|h_n\|^2 + \|g_n\|^2 \\ &\le C \int_{-\infty}^{\infty} \left(|h_n|^{q+2} + |g_n|^{p+2} + |h_n|^2 |g_n|\right) \; dx + C \left(1 + \|(h_n, g_n)\|_Y\right). \end{aligned} \tag{3.2}$$

From (3.1) it follows that

$$\begin{aligned} \int_{-\infty}^{\infty} |g_n|^{p+2} \; dx &\le C \|g_{nx}\|^{p/2} \|g_n\|^{(p+4)/2} \\ &\le C \left(\|(h_n, g_n)\|_Y^{p/2} + \|(h_n, g_n)\|_Y^{(3p+4)/4} \right). \end{aligned}$$

On the other hand, as in (2.2), we have

$$\int_{-\infty}^{\infty} |h_n|^{q+2} \; dx \le C \|h_{nx}\|^{q/2} \|h_n\|^{(q+4)/2} \le C \|(h_n, g_n)\|_Y^{q/2},$$

and, as in (2.4),

$$\int_{-\infty}^{\infty} |h_n|^2 |g_n| \; dx \le C \|h_{nx}\|^{1/2} \|g_n\| \le C \left(1 + \|(h_n, g_n)\|_Y\right).$$

Combining these estimates with (3.2) gives

$$\begin{aligned} &\|(h_n, g_n)\|_Y^2 \\ &\quad \le C \left(1 + \|(h_n, g_n)\|_Y + \|(h_n, g_n)\|_Y^{q/2} + \|(h_n, g_n)\|_Y^{p/2} + \|(h_n, g_n)\|_Y^{(3p+4)/4}\right), \end{aligned}$$

and since $q < 4$ and $p < 4/3$, the exponents on the right-hand side are all less than 2. Hence $\|(h_n, g_n)\|_Y$ is bounded. □

We omit the proofs of the next two lemmas, which are identical to the proofs of Lemmas 4.2 and 4.3 in [2].

Lemma 3.2. *Suppose $k, \theta \in \mathbb{R}$ and $h \in H^1_{\mathbb{C}}$. If $f(x) = e^{i(kx+\theta)} h(x)$, then*

$$E(f, g) = E(h, g) + k^2 H(h) - 2k \operatorname{Im} \int_{-\infty}^{\infty} h \overline{h}_x \; dx$$

and

$$G(f, g) = G(h, g) - kH(h).$$

Lemma 3.3. *Suppose $s > 0$ and $t \in \mathbb{R}$, and define $b = b(a) = (t-a)/s$ for $a \geq 0$. Then*

$$W(s,t) = \inf_{a \geq 0} \{I(s,a) + b(a)^2 s\}.$$

Lemma 3.4. *Suppose $s > 0$ and $t \in \mathbb{R}$, and define $b(a) = (t-a)/s$ for $a \geq 0$. If $\{(h_n, g_n)\}$ is a minimizing sequence for $W(s,t)$, then there exists a subsequence (still denoted by $\{(h_n, g_n)\}$) and a number $a \geq 0$ such that*

$$\lim_{n\to\infty} \|g_n\|^2 = a,$$

$$\lim_{n\to\infty} E(e^{ib(a)x} h_n, g_n) = I(s,a),$$

and

$$W(s,t) = I(s,a) + b(a)^2 s. \tag{3.3}$$

If $\beta_1 = 0$, we can further assert that $a > 0$.

Proof. The sequence a_n defined by

$$a_n = \|g_n\|^2 = G(h_n, g_n) - \mathrm{Im} \int_{-\infty}^{\infty} h_n \overline{h_{nx}}\, dx = t - \mathrm{Im} \int_{-\infty}^{\infty} h_n \overline{h_{nx}}\, dx$$

is bounded, by Lemma 3.1. Hence, by passing to a subsequence, we may assume that a_n converges to a limit $a \geq 0$. Let $b = b(a)$ and define $f_n(x) = e^{ibx} h_n(x)$. Then from Lemmas 3.2 and 3.3 we have that

$$\begin{aligned} \lim_{n\to\infty} E(f_n, g_n) &= \lim_{n\to\infty} \left(E(h_n, g_n) + b^2 H(h_n) - 2b\, \mathrm{Im} \int_{-\infty}^{\infty} h_n \overline{h_{nx}}\, dx \right) \\ &= W(s,t) + b^2 s - 2b(t-a) = W(s,t) - b^2 s \leq I(s,a). \end{aligned} \tag{3.4}$$

We claim that also

$$\lim_{n\to\infty} E(f_n, g_n) \geq I(s,a). \tag{3.5}$$

For if $a > 0$, then for sufficiently large n we have that $\|f_n\| > 0$ and $\|g_n\| > 0$, so the sequences $\beta_n = \sqrt{s}/\|f_n\|$ and $\theta_n = \sqrt{a}/\|g_n\|$ are defined, and both approach 1 as $n \to \infty$. Since $\|\beta_n f_n\|^2 = s$ and $\|\theta_n g_n\|^2 = a$, then $E(\beta_n f_n, \theta_n g_n) \geq I(s,a)$, and therefore

$$\lim_{n\to\infty} E(f_n, g_n) = \lim_{n\to\infty} E(\beta_n f_n, \theta_n g_n) \geq I(s,a).$$

On the other hand, if $a = 0$, then $\|g_n\| \to 0$ as $n \to \infty$, so it follows as in the proof of Lemma 2.2 that (2.6) holds: that is,

$$\lim_{n\to\infty} E(f_n, g_n) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1 |f_n|^{q+2} \right)\, dx \geq I(s,0). \tag{3.6}$$

Hence (3.5) holds in either case.

All the assertions of the Lemma, except the last one, now follow from (3.4) and (3.5).

To prove the last assertion of the Lemma, assume to the contrary that $\beta_1 = 0$ and $a = 0$. From Lemma 2.6 we know that $I(s,a) = 0$, so from

(3.3) it follows that $W(s,t) \geq 0$. But on the other hand, we can let g_0 be the function defined in Lemma 2.4, and f_0 be the corresponding function defined for this g_0 in Lemma 2.3. Then f_0 is real, $\|f_0\|^2 = s$, and $\|g_0\|^2 = t$, so $H(f_0) = s$ and $G(f_0, g_0) = t$, and hence $W(s,t) \leq E(f_0, g_0)$. Since

$$E(f_0, g_0) = \int_{-\infty}^{\infty} \left(f_{0x}^2 - \alpha f_0^2 g_0 \right)\, dx + J(g_0) < 0,$$

it follows that $W(s,t) < 0$, giving the desired contradiction. □

We can now prove Theorem 1.2. Statement (i) of the theorem is an immediate consequence of Lemma 3.4. To prove statement (ii), we start from a given subsequence and use Lemma 3.4 to conclude that some subsequence of $(f_n, g_n) = (e^{ibx} h_n, g_n)$ is a minimizing sequence for $I(s,a)$.

We claim that upon passing to a further subsequence, there exist real numbers y_n such that $(f_n(x + y_n), g_n(x + y_n))$ converges in Y to some (ϕ, ψ) in $\mathcal{S}_{s,a}$. If $a > 0$, this follows immediately from part (ii) of Theorem 1.1.

If, on the other hand, $a = 0$, then as in the proof of Lemma 3.4 we obtain (3.6). But from (3.6) we see that

$$\lim_{n\to\infty} E(f_n, g_n) = \lim_{n\to\infty} E(f_n, 0),$$

and since $E(f_n, g_n)$ converges to $I(s, 0)$, this means that $(f_n, 0)$ is a minimizing sequence for $I(s, 0)$. Since $a = 0$, then Lemma 3.4 implies that β_1 must be positive, so the claim follows from Lemma 2.5. Thus the claim has been proved in all cases.

Now, by passing to yet another subsequence, we may assume that e^{iby_n} converges to $e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Then $(h_n(. + y_n), g_n(. + y_n))$ converges to (Φ, ψ) in Y, where $\Phi(x) = e^{-i(bx+\theta)}\phi(x)$. As in (3.4), we have

$$\begin{aligned} I(s,a) = E(\phi, \psi) &= E(\Phi, \psi) + b^2 H(\Phi) - 2b\ \operatorname{Im} \int_{-\infty}^{\infty} \Phi \overline{\Phi}_x\, dx \\ &= E(\Phi, \psi) + b^2 s - 2b\left(G(\Phi, \psi) - \|\psi\|^2 \right) \\ &= E(\Phi, \psi) + b^2 s - 2b(t - s) = E(\Phi, \psi) - b^2 s. \end{aligned} \tag{3.7}$$

It then follows from (3.3) that $E(\Phi, \psi) = W(s,t)$, and hence that $(\Phi, \psi) \in \mathcal{F}_{s,t}$.

Part (iii) of the Theorem follows from the Lagrange multiplier principle, just as did part (iii) of Theorem 1.1.

Next we prove part (iv) of Theorem 1.2. Suppose $(\Phi, \psi) \in \mathcal{F}_{s,t}$. Applying Lemma 3.4 to the minimizing sequence for $W(s,t)$ defined by setting $(h_n, g_n) = (\Phi, \psi)$ for all $n \in \mathbb{N}$, we obtain that $(e^{ibx}\Phi, \psi)$ is a minimizing sequence for $I(s,a)$, where $a = \|g\|^2$ and $b = (t-a)/s$. Therefore $(e^{ibx}\Phi, \psi) \in \mathcal{S}_{s,a}$. Hence by part (iv) of Theorem 1.1, there exist a number $\theta \in \mathbb{R}$ and a real-valued function $\tilde{\phi}$ such that $e^{ibx}\Phi(x) = e^{i\theta}\tilde{\phi}(x)$. So

$$\Phi(x) = e^{i(-bx+\theta)}\tilde{\phi}(x),$$

which is (1.15). In case $\tau_1 = 0$, then $\beta_1 = 0$ and it follows from Lemma 3.4 that $a > 0$. Since $(\tilde{\phi}, \psi) \in \mathcal{S}_{s,a}$, it follows from part (iv) of Theorem 1.1 that $\psi(x) > 0$ on $\mathbb{R}$, and that either $\tilde{\phi}(x) > 0$ for all $x \in \mathbb{R}$ or $\tilde{\phi}(x) < 0$ for all $x \in \mathbb{R}$. In the latter case, we can add π to the value of θ and replace $\tilde{\phi}$ by $e^{i\theta}\tilde{\phi}$ to get that $\tilde{\phi}$ is positive on $\mathbb{R}$.

Part (v) of Theorem (1.2), concerning the stability of $\mathcal{F}_{s,t}$, follows from part (ii) by a standard argument, which we repeat here for completeness.

Suppose that $\mathcal{F}_{s,t}$ is not stable. Then there exist a number $\epsilon > 0$ and sequences (h_n, g_n) of initial data in Y and times $t_n \geq 0$ such that, for all $n \in \mathbb{N}$,

$$\inf\{\|(h_n, g_n) - (h, g)\|_Y : (h, g) \in \mathcal{F}_{s,t}\} < \frac{1}{n}; \tag{3.8}$$

while the solutions $(u_n(x,t), v_n(x,t))$ of (1.3) with initial data

$$(u_n(x,0), v_n(x,0)) = (h_n(x), g_n(x))$$

satisfy

$$\inf\{\|(u_n(\cdot, t_n), v_n(\cdot, t_n) - (h, g)\|_Y : (h, g) \in \mathcal{F}_{s,t}\} \geq \epsilon \tag{3.9}$$

for all $n \in \mathbb{N}$.

From (3.8) and Lemma 2.11 we have that

$$\begin{aligned} \lim_{n\to\infty} E(h_n, g_n) &= W(s,t), \\ \lim_{n\to\infty} H(h_n) &= s, \\ \lim_{n\to\infty} G(h_n, g_n) &= t. \end{aligned} \tag{3.10}$$

Let us denote $u_n(\cdot, t_n)$ by U_n and $v_n(\cdot, t_n)$ by V_n. Since $E(u,v)$, $G(u,v)$, and $H(u)$ are independent of t, then (3.10) implies

$$\begin{aligned} \lim_{n\to\infty} E(U_n, V_n) &= W(s,t), \\ \lim_{n\to\infty} H(U_n) &= s, \\ \lim_{n\to\infty} G(U_n, V_n) &= t, \end{aligned}$$

which means that $\{(U_n, V_n)\}$, like $\{(h_n, g_n)\}$, is a minimizing sequence for $W(s,t)$.

Now part (ii) of Theorem 1.2 tells us that there exists a subsequence $\{(U_{n_k}, V_{n_k})\}$, a sequence of real numbers $\{y_k\}$, and a function pair $(\Phi, \psi) \in \mathcal{F}_{s,t}$ such that

$$\lim_{k\to\infty} \|(U_{n_k}(\cdot + y_k), V_{n_k}(\cdot + y_k)) - (\Phi, \psi)\|_Y = 0. \tag{3.11}$$

So, for some sufficiently large k,

$$\|(U_{n_k}(\cdot + y_k), V_{n_k}(\cdot + y_k)) - (\Psi, \psi)\|_Y < \epsilon,$$

and hence

$$\|(U_{n_k}, V_{n_k}) - (\Phi(\cdot - y_k), \psi(\cdot - y_k))\|_Y < \epsilon. \tag{3.12}$$

But $(\Phi(\cdot - y_k), \psi(\cdot - y_k))$ is also in $\mathcal{F}_{s,t}$, and hence (3.12) gives

$$\inf\{\|(U_{n_k}, V_{n_k}) - (h, g)\|_Y : (h, g) \in \mathcal{F}_{s,t}\} < \epsilon.$$

Since this contradicts (3.9), we conclude that $\mathcal{F}_{s,t}$ must in fact be stable.

It remains only to prove the last assertion (vi) of Theorem 1.2, namely, that the sets $\mathcal{F}_{s,t}$ form a true two-parameter family. Suppose $(\Phi_1, \psi_1) \in \mathcal{F}_{s_1,t_1}$ and $(\Phi_2, \psi_2) \in \mathcal{F}_{s_2,t_2}$, where $(s_1, t_1) \neq (s_2, t_2)$. We want to show $(\Phi_1, \psi_1) \neq (\Phi_2, \psi_2)$. If $s_1 \neq s_2$, the conclusion is obvious, since then $\|\Phi_1\|^2 \neq \|\Phi_2\|^2$. So we can assume $s_1 = s_2$ and $t_1 \neq t_2$. From part (iv), if we let $\eta_i = (\|\psi_i\|^2 - t_1)/s_i$ for $i = 1, 2$; then there exist numbers θ_1 and θ_2 and real-valued functions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ such that

$$\Phi_1(x) = e^{i(\eta_1 x + \theta_1)}\tilde{\phi}_1(x) \quad \text{and} \quad \Phi_2(x) = e^{i(\eta_2 x + \theta_2)}\tilde{\phi}_2(x) \tag{3.13}$$

on $\mathbb{R}$. We may assume that $\Phi_1 = \Phi_2$, or else we are done. Then

$$e^{i((\eta_1 - \eta_2)x + (\theta_1 - \theta_2))} = \tilde{\phi}_2(x)/\tilde{\phi}_1(x)$$

is real-valued on $\mathbb{R}$, and hence η_1 must equal η_2. Since $s_1 = s_2$, this implies that $\|\psi_1\|^2 - t_1 = \|\psi_2\|^2 - t_2$. But $t_1 \neq t_2$, so therefore $\|\psi_1\|^2 \neq \|\psi_2\|^2$, and hence $\psi_1 \neq \psi_2$, as desired.

The proof of Theorem 1.2 is now complete.

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