STABILITY OF SOLITARY-WAVE SOLUTIONS TO LONG-WAVE EQUATIONS WITH GENERAL DISPERSION

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1. Introduction

In this report we consider nonlinear dispersive systems of the form

$$u_t + D(f(u) - Mu)_x = 0, \qquad (1)$$

where $u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$ is a map from $\mathbf{R} \times \mathbf{R}$ to $\mathbf{R}^n$, D is a constant diagonal matrix with positive entries, $f : \mathbf{R}^n \to \mathbf{R}^n$ is nonlinear, and the dispersion operator M acts as a Fourier multiplier operator in the x variable. More precisely, for each fixed t, Mu is a vector-valued function defined by

$$\widehat{Mu}(k,t) = m(k)\widehat{u}(k,t), \qquad k \in \mathbf{R},$$

where circumflexes denote Fourier transforms with respect to x,

$$\widehat{g}(k) = \int_{\mathbf{R}} e^{-ikx} g(x)\, dx.$$

The function $m(k)$, which takes values in the space of $n \times n$ matrices, is called the *multiplier* or *symbol* of M. In what follows we will assume that the entries of $m(k)$ are real, that $m(k) = m(-k)$ for all k, and that for each k the matrix $m(k)$ is symmetric. In particular, the first two of these assumptions guarantee that M takes $\mathbf{R}^n$-valued functions to $\mathbf{R}^n$-valued functions.

Equation (1), being the simplest possible form for an equation combining nonlinearity and dispersion, naturally arises as a model in many different physical contexts. One context which accounts for a variety of different dispersion operators is that of unidirectional internal gravity waves in two-dimensional inviscid stratified fluids between flat horizontal boundaries. Starting from the full Euler equations of motion governing such fluids, and performing a multi-scale analysis with wave amplitude and inverse wavelength taken as small parameters, one obtains at the first order of approximation a linear system which can be solved by separation of variables. If x denotes the horizontal coordinate and y the vertical coordinate, the separated solutions or *modes* are of the form $u(x,t)v(y)$, where $u(x,t)$ satisfies the ordinary one-dimensional linear wave equation, and $v(y)$ solves a Sturm-Liouville eigenvalue problem. Letting $v_1, v_2, \ldots$ denote the eigenfunctions of the Sturm-Liouville problem, one can write the general solution of the linear system as a superposition of modes $\sum u_i(x,t)v_i(y)$. At higher orders of approximation, modes can interact with each other, and the effects of the interaction on the functions $u_i(x,t)$ can in certain situations be modeled by a system of the form (1). A typical example is the Liu-Kubota-Ko system [24], considered in Section 2 below, for which $n = 2$ and each of the two modes in question represent disturbances concentrated at one of the two interfaces between three finite-depth layers of inviscid fluids with different densities. From the derivation given in [24] it is easy to see how to obtain other model equations of type (1) for different configurations, such as a system with more than two fluids, or with layers of infinite depth, or in which surface tension plays a significant role (see Section 3 below).

Of particular mathematical and physical interest are internal waves which propagate without spreading and with undiminished amplitude. Experiments and observations have suggested that such waves exist and maintain their form nearly unchanged on long time scales, either as one of a train of periodic waves or as a single solitary wave (see, e.g., [19, 21, 26, 27]). The corresponding mathematical phenomena, traveling-wave solutions of the governing equations of motion or of simplified model equations such as (1), have been intensively studied. For the full Euler equations for internal waves, there are a number of results on the existence of traveling-wave solutions (see, e.g., [8, 9, 14]) but it is not known whether these solutions are stable: in fact it is not known whether the Euler equations themselves are globally well-posed. (An interesting related result on *local* well-posedness appears in [29].) For equations of type (1), however, it has been possible to prove the existence of stable solitary-wave solutions, as well as global well-posedness of the general initial-value problem (see [1] for a brief overview of the extensive literature on this topic).

Previously obtained results on the stability of solitary-wave solutions of type (1) have dealt only with the case $n = 1$, and mostly with the case in which the symbol $m(k)$ of the dispersion operator takes its minimum value at $k = 0$. (This type of dispersion symbol seems to be generally associated with the existence of a positive solitary-wave solution.) In this paper we describe recent progress on extending the stability theory of solitary waves to cases in which $n > 1$, or in which $n = 1$ and $m(k)$ may take a minimum at a non-zero value of k. In Section 2, after defining more precisely the notion of stability we shall use, we announce a result giving sufficient conditions for the existence of stable solitary-wave solutions of (1); as a corollary we deduce the existence of stable (sets of) solitary waves for the Liu-Kubota-Ko system. In Section 3 we prove a stability result for scalar equations of type (1) for a broad class of symbols $m(k)$, with no restriction on the location of the minimum value of $m(k)$. The proof is essentially a generalization of J. Angulo's proof of stability in [10] for solitary-wave solutions of the Benjamin equation, in which $m(k) = \beta k^2 - \alpha|k|$ with $\alpha, \beta > 0$.

2. Sufficient conditions for stability of solitary waves

By a *solitary-wave* solution of (1) we mean a localized traveling-wave solution. More precisely, we define $u(x,t)$ to be a solitary-wave solution of (1) if it is of the form $u(x,t) = \phi(x - Ct)$, where $C \in \mathbf{R}$ and $\phi = (\phi_1, \ldots, \phi_n)$ with $\phi_i \in L^2(\mathbf{R})$ for $i = 1, \ldots, n$. Such a solitary wave is said to be *stable* if, whenever one solves (1) with initial data $\tilde{u}(x,0)$ which is a small perturbation of ϕ, the solution $\tilde{u}(x,t)$ remains close to the unperturbed solution $\phi(x - Ct)$ for all time.

Different notions of what it means for the perturbed solution to "remain close" to the unperturbed solution $\phi(x - Ct)$ lead to different definitions of stability. First of all, one must specify what it means for two functions of x to be close to each other: we will measure distance between $u(x)$ and $\tilde{u}(x)$ by the norm $\|u - \tilde{u}\|_X$ in the L^2-based Sobolev space $X = (H^s)^n = H^s \times \cdots \times H^s$. (Here standard notation is used: H^s is the set of distributions on $\mathbf{R}$ whose generalized derivatives up to order s are in $L^2(\mathbf{R})$. In general, the appropriate value of s for a stability result will depend on the properties of the functions $m(k)$ and $f(u)$ appearing in (1).) But it cannot be true in general that the perturbed solution $\tilde{u}(x,t)$ stays close in X norm to $\phi(x - Ct)$ for all time: to see this it suffices to consider $\tilde{u}(x,t) = \phi(x - \tilde{C}t)$, where $\tilde{C}$ is close to but not equal to C. In fact, this counterexample shows that the best possible stability result is that $\tilde{u}(x,t)$ stays close to the set of translations of the function $\phi(x)$, or in other words to the *orbit* of the solitary wave $\phi(x - Ct)$. This sort of stability is called orbital stability, and has indeed been proved to hold for a number of equations of type (1) (see, e.g., [2, 11, 18, 28]).

In this paper we use a different and (possibly) weaker notion of stability: namely, that of a set of solitary-wave profiles which does not necessarily constitute an orbit. Let $G \subseteq X$ be a set of vectors of solitary-wave profiles $\phi = (\phi_1, \ldots, \phi_n)$; i.e., each $\phi \in G$ corresponds to a solution $u(x,t) = \phi(x - Ct)$ of (1). (Typically, the value of the wavespeed C will be the same for all functions ϕ in G, although the theory does not require this to be the case.) We define G to be a stable set of solitary-wave profiles if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every ψ (in a suitable space Y of initial data) satisfying

$$\inf_{g \in G} \|\psi - g\|_X < \delta,$$

the solution $u(x,t)$ of (1) with initial data $u(x,0) = \psi(x)$ satisfies

$$\inf_{g \in G} \|u(x,t) - g\|_X < \epsilon \quad \text{for all} \quad t \in \mathbf{R}.$$

In some cases it is known that for a given wavespeed C, solitary-wave solutions of (1) are unique up to translation [6, 7]. In such cases, any set G of profiles of solitary waves with wavespeed C can consist only of translates of a single solitary-wave profile ϕ, and hence the existence of a stable set G of solitary waves with wavespeed C is equivalent to an orbital stability result for a single solitary-wave. If, on the other hand, two or more solitary-wave profiles exist which are not the same up to translation, then it might happen that a set G containing these two profiles is stable, without either of the individual profiles being orbitally stable. This distinction is relevant, for example, in the work of Maddocks and Sachs on stability of n-soliton solutions of the Korteweg-de Vries equation [25], where it is shown that the set of n-soliton solutions with specified wave-speeds $C_1, \ldots, C_n$ is stable, whereas an individual n-soliton solution in this set is not orbitally stable.

Implicit in the above definition of stability is the assumption that the initial-value problem for (1) is globally well-posed in some space Y of functions of x. By "globally well-posed in Y" we mean the following: for a given ψ in Y there exists a unique $u(x,t)$ such that $u(x,0) = \psi(x)$, $u(\cdot,t) \in Y$ for all $t \in \mathbf{R}$, and $u(x,t)$ is in some (possibly weak) sense a solution of (1). Moreover, the map from t to $u(\cdot,t)$ is in the space $C(\mathbf{R}, Y)$ of continuous maps from $\mathbf{R}$ to Y, and the correspondence $\psi \mapsto u(x,t)$ defines a continuous map from Y to $C(\mathbf{R}, Y)$. In what follows we assume that (1) is globally well-posed in some space Y which injects continuously into X; for global well-posedness results for some of the equations considered below we refer the reader to [5] and [22].

The approach to stability theory taken here will be the variational approach introduced by Cazenave and Lions [15, 16], which hinges on the properties of two functionals E and Q defined for $\psi \in X$ by

$$E(\psi) = \int_{-\infty}^{\infty} \frac{1}{2}\langle \psi, M\psi \rangle - F(\psi)\ dx$$

and

$$Q(\psi) = \int_{-\infty}^{\infty} \frac{1}{2}\langle \psi, D^{-1}\psi \rangle\ dx.$$

Here the brackets $\langle\ ,\ \rangle$ denote the usual inner products of vectors in $\mathbf{R}^n$, and $F : \mathbf{R}^n \to \mathbf{R}^n$ is defined by the relations $F' = f$ and $F(0) = 0$. Using the well-posedness assumptions for (1) mentioned above, one can show that E and

Q are conserved along solutions of (1); i.e., if $u(x,t)$ is any solution of (1) then $E(u(x,t)) = E(u(x,0))$ and $Q(u(x,t)) = Q(u(x,0))$ for all $t \in \mathbf{R}$. (For smooth solutions u, the invariance of $E(u)$ and $Q(u)$ may be established by multiplying (1) by Mu and by u, respectively, and performing appropriate integrations by parts. For general solutions in $(H^s)^n$, the result is derived from that for smooth solutions by approximating $u(x,0)$ by smooth initial data and using the assumption that solutions depend continuously on their initial data.)

Consider now the following constrained minimization problem: for given $q > 0$, find $g \in (H^s)^n$ such that $Q(g) = q$ and $E(g) = I_q$, where

$$I_q = \inf \{E(\psi) : \psi \in (H^s)^n \quad \text{and} \quad Q(\psi) = q\}.$$

The Lagrange multiplier equation $\delta E(g) = \lambda\ \delta Q(g)$ associated with this variational problem is in fact the equation which g must satisfy in order to that $u = g(x + \lambda t)$ be a solution of the system (1). Hence the set G_q of all solutions of the variational problem consists (if it is non-empty) of solitary-wave profiles for (1). The following theorem gives sufficient conditions for G_q to be stable.

Theorem 1. *Suppose that D is a diagonal matrix with positive entries, and $f(u) = (\alpha_1 |u_1|^{p+1}, \ldots, \alpha_n |u_n|^{p+1})$ where $\alpha_1, \ldots, \alpha_n$ are positive and $p > 1$. (If p is an integer we can also allow $f(u) = (\alpha_1 (u_1)^{p+1}, \ldots, \alpha_n (u_n)^{p+1})$.) Suppose that the $n \times n$ matrix-valued function $m(k) = \{m_{ij}(k)\}$ is an even function of k, with $m_{ij}(k)$ real for all i and j, and $m_{ij}(k) = m_{ji}(k)$ for $i \neq j$. Suppose further that $m(k)$ has the following three properties:*

1. *For all $k \in \mathbf{R}$, $m(k)$ is positive semi-definite; i.e., $\langle v, m(k)v \rangle \geq 0$ for all vectors $v \in \mathbf{R}^n$.*
2. *There exist positive constants A_1, A_2 and a number $s > p/4$ such that*

$$A_1 \langle v, v \rangle |k|^{2s} \leq \langle m(k)v, v \rangle \leq A_2 \langle v, v \rangle |k|^{2s}$$

holds for all vectors v in $\mathbf{R}^n$ and all sufficiently large values of $|k|$.

3. *For each i and j between 1 and n, the matrix components $m_{ij}(k)$ are infinitely differentiable functions of k at all $k \neq 0$. Moreover, for all*

$q \in \{0, 1, 2, \dots\}$ there exist constants $B = B_{ijq}$ and $K = K_{ijq}$ such that

$$\left| \left(\frac{d}{dk}\right)^q \left(\frac{m_{ij}(k) - m_{ij}(0)}{k}\right) \right| \le B|k|^{-q} \quad \text{for} \quad 0 < |k| \le K,$$

and

$$\left| \left(\frac{d}{dk}\right)^q \left(\frac{\sqrt{|m_{ij}(k)|}}{k^s}\right) \right| \le B|k|^{-q} \quad \text{for} \quad |k| \ge K.$$

If the solution I_q of the variational problem defined above satisfies $I_q < 0$ for all $q > 0$, then the set G_q of minimizers for the variational problem is non-empty, and is a stable set of solitary-wave solutions of (1) in the sense defined above.

Theorem 1 is a generalization of Theorem 4.1 of [1], which was stated for scalar equations only (i.e., for $n = 1$). Details of the proof will appear elsewhere; here we only display the lemma which constitutes the major step in the argument.

Lemma 2. *Under the assumptions of Theorem 1 (including the assumption that I_q is negative), the minimization problem for I_q enjoys the following compactness property: if $\{\psi_k\}_{k=1,2,\dots}$ is any minimizing sequence (i.e., $\psi_k \in (H^s)^n$ and $Q(\psi_k) \to q$ and $E(\psi_k) \to I_q$ as $k \to \infty$), then some subsequence of $\{\psi_k\}$ converges strongly in $(H^s)^n$ to a function g (which must therefore be in G_q).*

The Lemma is proved using Lions' method of *concentration compactness* [23]. As was originally observed by Cazenave and Lions [16], the stability of G_q follows from the compactness property proved in the Lemma by a simple argument in which E and Q play the role of Lyapunov functionals.

To illustrate the use of Theorem 1, we will apply it to the Liu-Kubota-Ko (LKK) system modeling the evolution and interaction of long, weakly nonlinear gravity waves propagating along the two interfaces separating three fluids of different densities [4, 24]. Again, we give here only an outline of the complete argument, deferring the details to an upcoming publication.

When physical variables are suitably scaled, the LKK system can be written in the form (1) with $n = 2$, $f(u) = (\alpha_1(u_1)^2, \alpha_2(u_2)^2)$,

$$D = \begin{pmatrix} 1/\gamma_4 & 0 \\ 0 & 1/\gamma_2 \end{pmatrix},$$

and

$$m(k) = \begin{pmatrix} m_{11}(k) & m_{12}(k) \\ m_{21}(k) & m_{22}(k) \end{pmatrix},$$

where

$$m_{11}(k) = \gamma_4 \left[c_1 + \gamma_1(k \coth kH_1 - 1/H_1) + \gamma_2(k \coth kH_2 - 1/H_2) \right],$$

$$m_{22}(k) = \gamma_2 \left[c_2 + \gamma_3(k \coth kH_3 - 1/H_3) + \gamma_4(k \coth kH_2 - 1/H_2) \right],$$

and

$$m_{12}(k) = m_{21}(k) = -\gamma_2\gamma_4 \left(\frac{k}{\sinh kH_2} \right).$$

The constants α_i, γ_i, c_i, and H_i are determined by the densities and depths of the three fluid layers.

From the derivation of the LKK system as a model equation, one sees that the γ_i (for $i = 1$ to 4) and the H_i (for $i = 1$ to 3) may always be taken to be positive. However, once this convention has been agreed upon, the signs of the constants α_1 and α_2 are then determined by the physical configuration of the system, and may be positive or negative, depending on the vertical structure of the modes corresponding to u_1 and u_2. For the purpose of applying Theorem 1 we assume that α_1 and α_2 are positive. (This assumption is valid, for example, if the modes are such that for every value of x and t, all the fluid particles in the corresponding vertical column of fluid have vertical velocities with the same sign [4].)

Next, observe that g is a solution of the problem of minimizing E subject to $Q = q$ if and only if g solves the problem of minimizing $\tilde{E} = E + CQ$ subject to $Q = q$, where C is an arbitrary constant. Moreover, the minimizing sequences for one problem are also the minimizing sequences for the other. Therefore, both problems simultaneously either enjoy or do not enjoy the compactness property described in Lemma 2, and so to prove stability of G_q it will suffice to show that the assumptions of Theorem 1 apply to the latter variational problem (for some C). Since

$$\tilde{E}(\psi) = \int_{-\infty}^{\infty} \frac{1}{2}\langle \psi, \tilde{M}\psi\rangle - F(\psi)\ dx,$$

where $\tilde{M} = CI + M$ and I is the $n \times n$ identity matrix, this amounts to showing that the function $\tilde{m}(k) = CI + m(k)$ satisfies assumptions 1, 2, and 3 of Theorem 1 (for some $s > p/4 = 1/4$), and that $\tilde{I}_q < 0$, where

$$\tilde{I}_q = \inf\ \{\tilde{E}(\psi) : \psi \in (H^s)^n \quad \text{and} \quad Q(\psi) = q\}.$$

Now a calculation of the eigenvalues $\beta_1(k)$ and $\beta_2(k)$ of $\tilde{m}(k)$ shows that there exists a unique C such that $0 \leq \beta_1(k) < \beta_2(k)$ for all $k \in \mathbf{R}$, with $\beta_1(k) = 0$ only when $k = 0$. For this C, it is easily verified that $\tilde{m}$ satisfies assumptions 1, 2, and 3 of Theorem 1, with $s = 1/2$. Moreover, taking $w(x) = v_0\psi(x)$, where $\psi(x)$ is smooth with compact support and v_0 is an appropriately normalized eigenvector for the eigenvalue $\beta_1(0) = 0$, one finds that $w_\theta(x) = \sqrt{\theta}w(\theta x)$ satisfies $\tilde{E}(w_\theta) < 0$ and $Q(w_\theta) = q$ for all sufficiently small values of θ. This shows that $\tilde{I}_q < 0$, so completing the proof that G_q is stable.

3. Scalar equations with general dispersion symbols

The Benjamin equation,

$$u_t + uu_x + \alpha\mathcal{H}u_{xx} + \beta u_{xxx} = 0, \tag{2}$$

was derived by Benjamin in [12] as a model for long, weakly nonlinear waves at the interface between two fluids of differing densities, in cases where one of the fluids has depth much greater than the other, and the surface tension at the interface is large enough to produce dispersive effects of the same order as finite-amplitude effects. (See [3] for more information concerning the physical assumptions underlying the derivation of the Benjamin equation.) In (2), the parameters α and β are positive constants and $\mathcal{H}$ denotes the Hilbert transform, which is by definition the Fourier multiplier operator with symbol $-i$ sign k.

Therefore, (2) can be written in the form (1), with $n = 1$, $f(u) = u^2/2$, and $m(k) = \beta k^2 - \alpha|k|$.

The solitary-wave solutions of (2) have been studied in [3, 10, 12, 13, 17]. In [13] Benjamin argues that solitary waves $u(x,t) = \phi(x - Ct)$ exist whenever C is such that the parameter $\gamma = \alpha/(2\sqrt{\beta C})$ lies in the range $0 < \gamma < 1$. He further uses formal computations to suggest that corresponding to a given $\gamma \in (0,1)$ there is a profile $\phi(x)$ which is not positive, but rather oscillates between positive and negative values finitely many times, with the number of oscillations increasing unboundedly as γ approaches 1. These conjectures were supported by numerical computations in [3], where a rigorous argument is also given for the existence of orbitally stable solitary waves for all sufficiently small values of γ. In [17], the existence of solitary waves is established rigorously for all $\gamma \in (0,1)$, but the variational characterization used there, which differs slightly from that of Cazenave and Lions, does not lead to a stability result. Recently, J. Angulo [10] has succeeded in applying the Cazenave-Lions method to prove existence and stability of solitary waves ϕ corresponding to all possible values $0 < q < \infty$ of the parameter $q = \int_{-\infty}^{\infty} \phi^2(x)\, dx$; thus showing in particular that stable solitary waves exist corresponding to values of γ arbitrarily close to 1.

The techniques used by Angulo in [10] make use of certain identities involving the Hilbert transforms of special functions, and so do not generalize immediately to equations with different dispersion operators. (See also [20], where similar techniques were used for equations in which the dispersion operator M is a partial differential operator.) We now show (see Theorem 2 below) how to adapt these techniques to obtain an existence and stability result for solitary-wave solutions for a broad class of equations of type (1) in the scalar case. The only restrictions put on the dispersion symbol $m(k)$ are that it be smooth and grow like a power of $|k|$ as $|k| \to \infty$; in particular we do not require, as in Theorem 1, that $m(k)$ be everywhere positive. In particular, besides recovering Angulo's stability result for the Benjamin equation, Theorem 2 applies as well to the finite-depth version of the Benjamin equation,

$$u_t + uu_x + Tu_x + u_{xxx} = 0,$$

in which T is the Fourier multiplier operator with symbol $m(k) = k \coth kH - 1/H$; and to the fifth-order equation

$$u_t + uu_x - u_{xxx} - \alpha u_{xxxxx} = 0 \qquad (\alpha > 0)$$

studied by Kichenassamy in [20].

For convenience we state the result for equations with the specific nonlinearity $f(u) = u^2/2$; it is clear that the proof generalizes easily to equations with other nonlinearities, but we do not pursue this question here.

Theorem 2. *Suppose $n = 1$ and $f(u) = u^2/2$ in equation (1); and suppose $m(k)$ is even and satisfies assumptions 2 and 3 of Theorem 1 for some $s > 1/4$. Then for every $q > 0$ the set G_q defined in Section 2 is non-empty, and consists of a stable set of solitary-wave profiles of (1).*

Proof. As was discussed above in Section 2, in proving the stability of G_q we may replace m by $\tilde{m} = m + C$ where C is an arbitrary constant; if the assumptions of Theorem 1 can be verified for $\tilde{m}$, then the stability of G_q will follow. Henceforth we will consider $\tilde{m}$ with $C = \min_{k \in \mathbf{R}} m(k)$, and tildes will be dropped for convenience of notation. From our assumptions it then follows that m satisfies assumptions 1, 2, and 3 of Theorem 1; moreover, there exists k_0 such that $m(k_0) = 0$. Since m is even, we may assume without loss of generality that $k_0 \geq 0$.

To complete the proof of the Theorem, it remains only to show that $I_q < 0$; for this we must find $\psi \in H^s(\mathbf{R})$ such that $Q(\psi) = q$ and $E(\psi) < 0$. We will adapt a construction used by Angulo in [10].

First we consider the case when $k_0 \neq 0$. Let $h(x) = 1/(1 + x^2)$, and define

$$\psi = ah(\epsilon x) \left(\cos(k_0 x) + \epsilon\right),$$

where $\epsilon = q^2$, and a is chosen so that $Q(\psi) = q = \sqrt{\epsilon}$. We will show that $E(\psi) < 0$ when ϵ and q are sufficiently small.

To describe the behavior of a as $\epsilon \to 0$, we use the double-angle formula for the cosine to write

$$\begin{aligned}\sqrt{\epsilon} &= \frac{a^2}{2}\int_{-\infty}^{\infty} h(\epsilon x)^2(\cos(k_0 x)+\epsilon)^2\,dx \\ &= \frac{a^2}{4\epsilon}\left[\left(1+2\epsilon^2\right)\left(\int_{-\infty}^{\infty} h(x)^2\,dx\right) + 4\epsilon\widehat{h^2}(k_0/\epsilon) + \widehat{h^2}(2k_0/\epsilon)\right].\end{aligned}$$

Since $\widehat{h^2}(k) = (\pi/2)(|k|+1)e^{-|k|}$, it follows that

$$a = \frac{\epsilon^{3/4}}{b+o(1)} \tag{3}$$

as $\epsilon \to 0$, where $b = \frac{1}{2}\left(\int_{-\infty}^{\infty} h(x)^2\,dx\right)^{1/2} > 0$. (Here and in what follows, we use the usual "little-o, big-O" notation, so that $o(\epsilon)$, for example, denotes a quantity which tends to zero faster than ϵ as $\epsilon \to 0$, while $O(\epsilon)$ denotes a quantity whose absolute value is dominated by a constant times ϵ as $\epsilon \to 0$.)

Next we investigate the behavior as $\epsilon \to 0$ of the quadratic part of $E(\psi)$, given by the integral

$$\begin{aligned}&\frac{1}{2}\int_{-\infty}^{\infty}\psi(x)M\psi(x)\,dx = \\ &\quad = \frac{a^2}{2}\int_{-\infty}^{\infty}\left[\widehat{h}(k/\epsilon) + \frac{1}{2\epsilon}\widehat{h}\left((k-k_0)/\epsilon\right) + \frac{1}{2\epsilon}\widehat{h}\left((k+k_0)/\epsilon\right)\right]^2 m(k)\,dk.\end{aligned} \tag{4}$$

Expanding the squared expression in brackets on the right-hand side of (4), we obtain six integrals which we will estimate separately.

First, write

$$\int_{-\infty}^{\infty}\widehat{h}(k/\epsilon)^2 m(k)\,dk = \epsilon\int_{-\infty}^{\infty}\widehat{h}(k)^2 m(k\epsilon)\,dk,$$

and apply the Dominated Convergence Theorem (note that $\widehat{h}(k) = \pi e^{-|k|}$, like $\widehat{h^2}(k)$, decays exponentially as $|k| \to \infty$) to obtain

$$\int_{-\infty}^{\infty}\widehat{h}(k/\epsilon)^2 m(k)\,dk = \epsilon\, m(0)\int_{-\infty}^{\infty}\widehat{h}(k)^2\,dk + o(\epsilon)$$

as $\epsilon \to 0$.

Next, write

$$\frac{1}{4\epsilon^2}\int_{-\infty}^{\infty} \widehat{h}\left((k-k_0)/\epsilon\right)^2 m(k)\ dk = \frac{1}{4\epsilon}\int_{-\infty}^{\infty} \widehat{h}(t)^2 m(k_0+\epsilon t)\ dt.$$

Since $m(k)$ is smooth, has a relative minimum at $k = k_0$ with $m(k_0) = 0$, and grows like $|k|^{2s}$ for $|k|$ large, then by Taylor's theorem there exists a constant A such that for all $r \in \mathbf{R}$,

$$m(k_0+r) = \left(\frac{m''(k_0)}{2}\right) r^2 + \mu(r),$$

where $\mu(r) \le A|r|^3$ for $|r| \le 1$ and $\mu(r) \le A|r|^{\max(2s,3)}$ for $|r| \ge 1$. It follows that

$$\frac{1}{4\epsilon^2}\int_{-\infty}^{\infty} \widehat{h}\left((k-k_0)/\epsilon\right)^2 m(k)\ dk = \epsilon\left(\frac{m''(k_0)}{8}\right)\int_{-\infty}^{\infty} \widehat{h}(t)^2 t^2\ dt + o(\epsilon).$$

A similar argument shows that we also have

$$\frac{1}{4\epsilon^2}\int_{-\infty}^{\infty} \widehat{h}\left((k+k_0)/\epsilon\right)^2 m(k)\ dk = \epsilon\left(\frac{m''(k_0)}{8}\right)\int_{-\infty}^{\infty} \widehat{h}(t)^2 t^2\ dt + o(\epsilon).$$

Finally, consider the integrals arising from the cross terms in (4), such as

$$\int_{-\infty}^{\infty} \widehat{h}\left((k+k_0)/\epsilon\right) \widehat{h}\left((k-k_0)/\epsilon\right) m(k)\ dk.$$

Splitting the interval of integration into the two intervals $-\infty < k \le 0$ and $0 \le k < \infty$, and noting that $\widehat{h}\left((k-k_0)/\epsilon\right) = \pi e^{-|k-k_0|/\epsilon} \le \pi e^{-k_0/\epsilon}$ for $k \le 0$ and $\widehat{h}\left((k+k_0)/\epsilon\right) \le \pi e^{-k_0/\epsilon}$ for $k \ge 0$, one sees that the integral goes to 0 faster than any power of ϵ as $\epsilon \to 0$. Similar arguments lead to the same estimate for the integrals $\int_{-\infty}^{\infty} \widehat{h}(k/\epsilon)\widehat{h}\left((k+k_0)/\epsilon\right) m(k)\ dk$ and $\int_{-\infty}^{\infty} \widehat{h}(k/\epsilon)\widehat{h}\left((k-k_0)/\epsilon\right) m(k)\ dk$. (For the latter integral, for example, we split the interval of integration into the subintervals $-\infty < k \le k_0/2$ and $k_0/2 \le k < \infty$.)

Combining the above estimates, we obtain finally

$$\frac{1}{2}\int_{-\infty}^{\infty} \psi(x) M\psi(x)\ dx = \frac{\epsilon^{5/2}}{2b^2+o(1)}\left[\int_{-\infty}^{\infty} (\widehat{h}(k))^2\left(m(0) + \frac{m''(k_0)k^2}{4}\right)\ dk + o(1)\right]$$

as $\epsilon \to 0$.

It remains to estimate the cubic part of $E(\psi)$. Using the double- and triple-angle identities for the cosine, we obtain

$$\int_{-\infty}^{\infty} \psi(x)^3 \, dx = \frac{a^3}{\epsilon} \int_{-\infty}^{\infty} h(x)^3 \left(\cos(k_0 x/\epsilon) + \epsilon\right)^3 \, dx$$
$$= \frac{a^3}{4\epsilon} \left[(6\epsilon + 4\epsilon^3) \int_{-\infty}^{\infty} h(x)^3 \, dx + (3 + 12\epsilon^2)\widehat{h^3}(k_0/\epsilon) + 6\epsilon \widehat{h^3}\,(2k_0/\epsilon) + \widehat{h^3}\,(3k_0/\epsilon) \right]$$
$$= a^3 \left[\frac{3}{2} \int_{-\infty}^{\infty} h(x)^3 \, dx + o(\epsilon) \right],$$

since $\widehat{h^3}(k) \to 0$ exponentially fast as $|k| \to \infty$. Hence, using (3), we conclude that there exists a positive constant A such that

$$\int_{-\infty}^{\infty} \psi(x)^3 \, dx \geq A\epsilon^{9/4}$$

for all sufficiently small values of ϵ.

Since $\int_{-\infty}^{\infty} \psi(x) M\psi(x) \, dx$ is dominated by a constant times $\epsilon^{5/2}$ as $\epsilon \to 0$, and $\int_{-\infty}^{\infty} \psi(x)^3 \, dx$ is greater than a positive constant times $\epsilon^{9/4}$, it follows that

$$E(\psi) = \frac{1}{2} \int_{-\infty}^{\infty} \psi(x) M\psi(x) \, dx - \frac{1}{6} \int_{-\infty}^{\infty} \psi(x)^3 \, dx$$

becomes negative when ϵ is sufficiently small. Thus it has been proved that there exists $q_0 > 0$ such that $I_q < 0$ for all $q \in (0, q_0]$.

It remains to show that in fact $I_q < 0$ for all $q \geq q_0$ as well. To see this, let ψ be the function constructed as above for $q = q_0$, so that $E(\psi) < 0$ and $Q(\psi) = q_0$. For given $q \geq q_0$, we define $\tilde{\psi} = \beta\psi$, where $\beta = \sqrt{q/q_0}$. Since $\beta \geq 1$ and $\int_{-\infty}^{\infty} \psi(x)^3 \, dx > 0$, then

$$E(\tilde{\psi}) = \frac{\beta^2}{2} \int_{-\infty}^{\infty} \psi(x) M\psi(x) \, dx - \frac{\beta^3}{6} \int_{-\infty}^{\infty} \psi(x)^3 \, dx$$
$$\leq \beta^2 \left(\frac{1}{2} \int_{-\infty}^{\infty} \psi(x) M\psi(x) \, dx - \frac{1}{6} \int_{-\infty}^{\infty} \psi(x)^3 \, dx \right) = \beta^2 E(\psi) < 0.$$

But $Q(\tilde{\psi}) = q$, and so it has been proved that $I_q < 0$. This completes the proof of Theorem 2 in the case $k_0 \neq 0$.

The case $k_0 = 0$ is simpler (and in fact has already been treated in [1]). Again, the issue is to show that for any $q > 0$ there exists a function $\psi \in H^s(\mathbf{R})$ such that $Q(\psi) = q$ and $E(\psi) < 0$.

We let $h(x)$ be an arbitrary smooth, positive function of compact support such that $Q(h) = q$, and define $\psi(x) = \sqrt{\epsilon} h(\epsilon x)$, where ϵ is to be determined below. Then for all ϵ, $Q(\psi) = q$, and the cubic part of $E(\psi)$ is given by

$$\int_{-\infty}^{\infty} \psi(x)^3 \, dx = \sqrt{\epsilon} \int_{-\infty}^{\infty} h(x)^3 \, dx.$$

To estimate the quadratic part of $E(\psi)$, we write

$$\int_{-\infty}^{\infty} \psi(x) M \psi(x) \, dx = \int_{-\infty}^{\infty} \widehat{h}(k)^2 m(\epsilon k) \, dk.$$

Since $m(k)$ satisfies assumption 2 of Theorem 1 (and $m(0) = 0$, because $k_0 = 0$), then there exists a constant $A > 0$ such that $|m(k)| \leq A|k|$ for $0 \leq |k| \leq 1$ and $|m(k)| \leq |k|^{\max(s,1)}$ for $|k| \geq 1$. It follows that

$$\int_{-\infty}^{\infty} \psi(x) M \psi(x) \, dx = O(\epsilon)$$

as $\epsilon \to 0$. Thus the quadratic part of E is of higher order than the cubic part, so taking ϵ sufficiently close to zero gives $E(\psi) < 0$, as desired. □

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