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Concentration Compactness and the Stability of Solitary-Wave Solutions to Nonlocal Equations

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Abstract. In their proof of the stability of standing-wave solutions of nonlinear Schrödinger equations, Cazenave and Lions used the principle of concentration compactness to characterize the standing waves as solutions of a certain variational problem. In this article we first review the techniques introduced by Cazenave and Lions, and then discuss their application to solitary-wave solutions of nonlocal nonlinear wave equations. As an example of such an application, we include a new result on the stability of solitary-wave solutions of the Kubota-Ko-Dobbs equation for internal waves in a stratified fluid.

1. Introduction

The first mathematical treatment of the problem of stability of solitary waves was published in 1871 by Joseph Boussinesq [**Bou**], who at the time was 29 years old and just beginning a long and distinguished career in mathematical physics. The solitary waves he was concerned with are water waves with readily recognizable hump-like profiles, which are often produced by disturbances in a shallow channel and which can undergo strong interactions and travel long distances without evident change in form.

Boussinesq showed in [**Bou**] that if a water wave propagates along a flat-bottomed channel of undisturbed depth H, and has large wavelength and small amplitude relative to H, then the elevation h of the water surface considered as a function of the coordinate x along the channel and the time t will approximately satisfy the equation

$$h_{tt} - gHh_{xx} - gH(\frac{3}{2H}h^2 + \frac{H^2}{3}h_{xx})_{xx} = 0,$$

where g is the gravitational acceleration. Using this equation he obtained an explicit representation of solitary waves in terms of elementary functions (reproduced below as equation (1.2)). He then proposed to show that solitary waves are stable in the sense that a slight perturbation of a solitary wave will continue to resemble a solitary wave for all time, rather than evolving into some other wave form. Such a result

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would go a long way towards explaining why solitary waves are so easily produced and observed in experiments.

Boussinesq's proof of stability centered upon the quantity

$$\int_{-\infty}^{\infty} \left\{ \left(\frac{dh}{dx}\right)^2 - \frac{3h^3}{H^3} \right\} \, dx,$$

which he called the *moment of instability*. Like a wave's volume $\int_{-\infty}^{\infty} h \, dx$ or its energy $\int_{-\infty}^{\infty} h^2 \, dx$, its moment of instability does not change from one instant to the next as the wave evolves. Boussinesq asserted that, within the class of wave profiles whose energy has a given value, those profiles which correspond to the greatest moments of instability will differ the most from solitary-wave profiles, while the minimum value of the moment of instability within this class is attained at a solitary-wave profile. It follows that solitary waves must be stable, for if a wave closely resembles a solitary wave at some time, then its moment of instability must be close to that of a solitary wave; but since the moment of instability does not change, then the wave must remain close to a solitary wave for all time ([**Bou**], p. 62).

In support of his assertions, Boussinesq gave an ingenious proof, which is still worthy of study today, that if energy is held constant then the moment of instability is minimized at a solitary wave. By modern standards, however, his overall argument for the stability of solitary waves contains some gaps: for example he does not explain why two functions whose moments of stability are nearly the same must resemble each other in form.

The first rigorous proof of stability of solitary waves appeared a century later, in Benjamin's article [**B2**] on solitary-wave solutions of the Korteweg-deVries equation

$$u_t + uu_x + u_{xxx} = 0. \tag{1.1}$$

(Actually the argument in [**B2**] still contained some flaws which were mended by Bona [**Bo**] a few years later.) Equation (1.1), a model equation for water waves derived in [**KdV**] some twenty years after Boussinesq published his work, has solutions $u(x,t) = \phi_C(x - Ct)$ which correspond to the solitary waves studied by Boussinesq: here the wavespeed C can be any positive number, and the wave profile ϕ_C is given as a function of its argument $\xi = x - Ct$ by

$$\phi_C(\xi) = \frac{3C}{\cosh^2(\frac{1}{2}\sqrt{C}\xi)}. \tag{1.2}$$

The quantities referred to by Boussinesq as the "energy" and the "moment of instability" of a water wave correspond to the functionals

$$Q(u) = \frac{1}{2}\int_{-\infty}^{\infty} u^2 \, dx \tag{1.3}$$

and

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty} \left[(u_x)^2 - \frac{1}{3}u^3 \right] \, dx, \tag{1.4}$$

respectively. It is an easy exercise to see that if $u = u(x,t)$ satisfies (1.1) then both of these functionals are independent of time.

In the setting of equation (1.1), Boussinesq's assertion would be that ϕ_C is a minimizer of the functional E over the set of all admissible functions ψ satisfying $Q(\psi) = Q(\phi_C)$. Benjamin did not prove this in its entirety, but did show that ϕ_C is a local minimizer. More precisely, he showed that if $\psi \in H^1$ is sufficiently close to ϕ_C in H^1 norm, and $Q(\psi) = Q(\phi_C)$, then

$$E(\psi) - E(\phi_C) \geq A \inf_{y \in \mathbf{R}} \|\psi - \phi_C(\cdot + y)\|_{H^1}$$

where A denotes a positive constant which is independent of ψ. From this estimate, together with an elaboration of Boussinesq's original argument, one can deduce the following stability result:

THEOREM 1.1 [**B2,Bo**]. *For every $\epsilon > 0$, there exists $\delta > 0$ such that if*

$$\|u_0 - \phi_C\|_{H^1} < \delta,$$

then the solution $u(x,t)$ of (1.1) with $u(x,0) = u_0$ satisfies

$$\inf_{y \in \mathbf{R}} \|u(x,t) - \phi_C(x+y)\|_{H^1(dx)} < \epsilon$$

for all $t \in \mathbf{R}$.

(In stating this theorem we have tacitly assumed that the initial data u_0 belong to a class for which unique solutions of the initial-value problem for (1.1) exist for all time. We will continue to make this assumption without comment in what follows; for more information on well-posedness of the initial value problem for (1.1) and related equations the reader may consult [**ABFS**].)

Note that in Theorem 1.1 it cannot be concluded that

$$\|u(x,t) - \phi_C(x - Ct)\|_{H^1(dx)} < \epsilon$$

for all time: to see this, it suffices to consider the solution $u(x,t) = \phi_{C_1}(x - C_1 t)$, where C_1 is close to, but not equal to C. However, Bona and Soyeur [**BS**] have recently shown that if $\|u_0 - \phi_C\|_1$ is sufficiently small then

$$\|u(x,t) - \phi_C(x - \gamma(t))\|_{H^1(dx)} < \epsilon$$

and

$$|\gamma'(t) - C| < \theta\epsilon$$

for all t, where the constant θ is independent of ϵ and u_0.

In [**B2**] Benjamin pointed out that Boussinesq's idea could also be applied to solitary wave solutions of other equations of Hamiltonian form. Others soon took up this suggestion and extended the theory to more general settings: see, e.g., the far-reaching treatments in [**GSS1,GSS2,MS**] (some of whose results were anticipated in the physics literature [**KRZ**]). Recently, Pego and Weinstein introduced a particularly interesting variation of the theory [**PW**], showing that solitary-wave solutions of the KdV equation (1.1) are asymptotically stable: i.e., a small initial perturbation of a solitary wave will give rise to a solution $u(x,t)$ which not only resembles a solitary wave for all time, but in fact tends to the solitary wave as a limiting form as $t \to \infty$. (Boussinesq, by the way, had already noted that this phenomenon appears in experiments, but had attributed it to frictional effects—which are not modelled by the inviscid equation (1.1).) In all these papers, the solitary wave is proved stable by showing that it is a local (as opposed to global) constrained minimizer of a Hamiltonian functional E; this is done by analyzing the functional derivatives of E and the constraint functional Q at the solitary wave.

An alternate approach to proving stability of solitary waves, which does not rely on local analysis, was developed by Cazenave and P. Lions [**C,CL**] using Lions' method of *concentration compactness*. In this approach, rather than starting with a given solitary wave and attempting to prove that it realizes a local minimum of a constrained variational problem, one starts instead with the constrained variational problem and looks for global minimizers. When the method works, it shows not only that global minimizers exist, but also that every minimizing sequence is relatively compact up to translations (cf. Theorem 2.9 below). An easy corollary is that the set of global minimizers is a stable set for the associated initial value problem, in the sense that a solution which starts near the set will remain near it for all time.

Although the concentration-compactness method for proving stability of solitary waves has the advantage of requiring less detailed analysis than the local methods, it also produces a weaker result in that it only demonstrates stability of a set of minimizing solutions without providing information on the structure of that set, or distinguishing among its different members. Thus, when de Bouard and Saut [**dBS**] used the concentration-compactness method to prove stability of a set of traveling-wave solutions of the Kadomtsev-Petviashvili equation, they noted that their result did not establish the stability of the explicit "lump" solitary-wave solutions of this equation, since it is not known whether the lump solutions are in the stable set. Further, even if the lump solutions were known to be in this set, the possibility remains open that there are other elements in the stable set which do not resemble lump solutions, and that a perturbed lump solution may wander towards these other minimizers.

In this paper we use concentration compactness to prove stability of solitary-wave solutions of equations of the form

$$u_t + (f(u))_x - (Lu)_x = 0, \tag{1.5}$$

where $f(u)$ is a real-valued function of u, and Lu is a Fourier multiplier operator defined by

$$\widehat{Lu}(k) = m(k)\widehat{u}(k)$$

(here the hats denote Fourier transforms). If $m(k)$, the *symbol* of L, is a polynomial function of k, then L is a differential operator; and in particular is a *local* operator in the sense that if $u = 0$ outside an open subset S of $\mathbf{R}$ then also $Lu = 0$ outside S. On the other hand, in many situations in fluid dynamics and mathematical physics, equations of the above type arise in which $m(k)$ is not a polynomial and hence the operator L is *nonlocal* (see Sections 3 and 4 below for examples).

The functionals

$$Q(u) = \frac{1}{2}\int_{-\infty}^{\infty} u^2 \, dx$$

and

$$E(u) = \int_{-\infty}^{\infty} \left[\frac{1}{2}uLu - F(u)\right] \, dx,$$

where $F'(u) = f(u)$ for $u \in \mathbf{R}$ and $F(0) = 0$, are integrals of motion for (1.5); i.e., if u solves (1.5) then $Q(u)$ and $E(u)$ are independent of t. The concentration-compactness method for proving stability of solitary-wave solutions of (1.5) aims to show that they are minimizers of E subject to the constraint that Q be held constant. In the original version of the method, as it appears for example in the article [**CL**] on nonlinear Schrödinger equations, important use is made of the fact that the operator L appearing in E is a local operator. Nevertheless, we show below (using an idea which has already appeared in [**ABS**]) that if L is not too far from being local then the results of [**CL**] still obtain. Of course, as explained above, the stability result in view is that the set of solitary waves which are solutions of the minimization problem is stable as a set; and to the extent that the structure of this set remains unknown, such a stability result leaves something to be desired.

The remainder of this paper is organized as follows. In Section 2, we review the concentration-compactness method by showing in detail how it is used to prove stability of solitary-wave solutions of the Korteweg-de Vries equation. The proof is broken into a series of short lemmas to make it easy to identify the parts which need to be modified in the nonlocal case. In Section 3, we illustrate how the method may be applied to nonlocal equations by using it to prove the stability of solitary-wave solutions of the Kubota-Ko-Dobbs equation for internal waves in stratified fluids. Generalizations of the result obtained in Section 3 are discussed in the concluding Section 4.

NOTATION. The set of natural numbers $\{0, 1, 2, \dots\}$ and the set of all integers are written $\mathbf{N}$ and $\mathbf{Z}$, respectively. The set of all real numbers is denoted by $\mathbf{R}$, and all integrals will be taken over $\mathbf{R}$ unless otherwise specified. The Fourier transform $\widehat{f}$ of a tempered distribution $f(x)$ on R is defined as

$$\widehat{f}(k) = \int e^{-ikx} f(x)\ dx.$$

For any tempered distribution f on $\mathbf{R}$ and any $s \in \mathbf{R}$, we define

$$\|f\|_s = \left(\int (1 + |k|^2)^{s/2} |\widehat{f}(k)|^2\ dk \right)^{1/2},$$

and H^s denotes the Sobolev space of all f for which $\|f\|_s$ is finite. The notation $l_2(H^s)$ will be used for the Hilbert space of all sequences $\{g_j\}_{j\in\mathbf{Z}}$ such that $g_j \in H^s$ for each j and $\sum_{j\in\mathbf{Z}} \|g_j\|_s^2 < \infty$. For any measurable function f on $\mathbf{R}$ and any $p \in [1, \infty)$, we define

$$|f|_p = \left(\int |f(x)|^p\ dx \right)^{1/p},$$

and L^p denotes the space of all f for which $|f|_p$ is finite. The space L^∞ is defined as the space of all measurable functions f on $\mathbf{R}$ such that

$$|f|_\infty = \operatorname*{ess\,sup}_{x\in\mathbf{R}} |f(x)|$$

is finite. Finally, if E is a subset of $\mathbf{R}$ then $C_0^\infty(E)$ denotes the space of infinitely differentiable functions with compact support in E.

2. The concentration-compactness method

In this section we illustrate the concentration-compactness method by using it to prove Benjamin and Bona's result (Theorem 1.1 above) on the stability of solitary-wave solutions of the KdV equation (1.1). We have broken the proof into small lemmas so that later, in Section 3, it will be easy to identify the parts which need changing. The ideas in this section are not new, but are rather a selection of arguments adapted from [**CL**] and [**L**] (see also [**C**]).

Let E and Q be as defined in (1.3) and (1.4); it is easy to see using the Sobolev embedding theorem that E and Q define continuous maps from H^1 to $\mathbf{R}$. Fix a positive number C and let ϕ_C be as in (1.2). Let $q = Q(\phi_C)$ and define the real number I_q by

$$I_q = \inf \left\{ \ E(\psi) : \psi \in H^1 \ \text{ and } \ Q(\psi) = q \ \right\}.$$

The *set of minimizers* for I_q is

$$G_q = \left\{ \ \psi \in H^1 : E(\psi) = I_q \ \text{ and } \ Q(\psi) = q \ \right\},$$

and a *minimizing sequence* for I_q is any sequence $\{f_n\}$ of functions in H^1 satisfying

$$Q(f_n) = q \quad \text{for all } n$$

and

$$\lim_{n \to \infty} E(f_n) = I_q.$$

To each minimizing sequence $\{f_n\}$, we associate a sequence of nondecreasing functions $M_n : [0, \infty) \to [0, q]$ defined by

$$M_n(r) = \sup_{y \in \mathbf{R}} \int_{y-r}^{y+r} |f_n|^2 \ dx.$$

An elementary argument shows that any uniformly bounded sequence of nondecreasing functions on $[0, \infty)$ must have a subsequence which converges pointwise (in fact, uniformly on compact sets) to a nondecreasing limit function on $[0, \infty)$. Hence $\{M_n\}$ has such a subsequence, which we denote again by $\{M_n\}$. Let $M : [0, \infty) \to [0, q]$ be the nondecreasing function to which M_n converges, and define

$$\alpha = \lim_{r \to \infty} M(r);$$

then $0 \leq \alpha \leq q$.

The method of concentration compactness [**L1,L2**], as applied to this situation, consists of two observations. The first is that if $\alpha = q$, then the minimizing sequence $\{f_n\}$ has a subsequence which, when its terms are suitably translated, converges strongly in H^1 to an element of G_q. The second is that certain simple properties of the variational problem imply that α must equal q for *every* minimizing sequence $\{f_n\}$. It follows that not only do minimizers exist in H^1, but every minimizing sequence converges in H^1 norm to the set G_q.

We will now give the details of the method. We begin with a special case of a general result found in [**L1**] and [**Li**].

Lemma 2.1. *Suppose $B > 0$ and $\delta > 0$ are given. Then there exists $\eta = \eta(B, \delta) > 0$ such that if $f \in H^1$ with $\|f\|_1 \leq B$ and $|f|_3 \geq \delta$, then*

$$\sup_{y \in \mathbf{R}} \int_{y-1/2}^{y+1/2} |f(x)|^3 \ dx \geq \eta.$$

Proof. We have

$$\sum_{j \in \mathbf{Z}} \int_{j-1/2}^{j+1/2} \left[f^2 + (f')^2\right] \ dx = \|f\|_1^2 \leq B^2 = \frac{B^2}{|f|_3^3} \ |f|_3^3 = \sum_{j \in \mathbf{Z}} \frac{B^2}{|f|_3^3} \ \int_{j-1/2}^{j+1/2} |f|^3 \ dx.$$

Hence there exists some $j_0 \in \mathbf{Z}$ for which

$$\int_{j_0-1/2}^{j_0+1/2} [f^2 + (f')^2]\ dx \le \frac{B^2}{|f|_3^3} \int_{j_0-1/2}^{j_0+1/2} |f|^3\ dx.$$

Now by the Sobolev embedding theorem there exists a constant A (independent of f) such that

$$\left(\int_{j_0-1/2}^{j_0+1/2} |f|^3\ dx\right)^{1/3} \le A \left(\int_{j_0-1/2}^{j_0+1/2} [f^2 + (f')^2]\ dx\right)^{1/2}.$$

Hence

$$\left(\int_{j_0-1/2}^{j_0+1/2} |f|^3\ dx\right)^{1/3} \le \left(\frac{AB}{|f|_3^{3/2}}\right) \left(\int_{j_0-1/2}^{j_0+1/2} |f|^3\ dx\right)^{1/2},$$

so

$$\int_{j_0-1/2}^{j_0+1/2} |f|^3\ dx \ge \frac{\delta^9}{A^6 B^6}.$$

The proof is now concluded by taking η to be the constant on the right-hand side of the last inequality. $\square$

Next we establish some properties of the variational problem and its minimizing sequences which are independent of the value of α.

Lemma 2.2. *For all $q_1 > 0$, one has*

$$-\infty < I_{q_1} < 0.$$

Proof. Choose any function $\psi \in H^1$ such that $Q(\psi) = q_1$ and $\int \psi^3\ dx \neq 0$. For each $\theta > 0$, define the function ψ_θ by $\psi_\theta(x) = \sqrt{\theta}\psi(\theta x)$. Then for all θ one has $Q(\psi_\theta) = Q(\psi) = q_1$ and

$$E(\psi_\theta) = \frac{\theta^2}{2} \int (\psi')^2\ dx - \frac{\theta^{1/2}}{6} \int \psi^3\ dx.$$

Hence by taking $\theta = \theta_0$ sufficiently small we get $E(\psi_{\theta_0}) < 0$, and since $I_{q_1} \le E(\psi_{\theta_0})$ it follows that $I_{q_1} < 0$.

Now let ψ denote an arbitrary function in H^1 satisfying $Q(\psi) = q_1$. To prove that $I_{q_1} > -\infty$, it suffices to bound $E(\psi)$ from below by a number which is independent of ψ. Note first that from standard Sobolev embedding and interpolation theorems it follows that

$$\left|\int \psi^3\ dx\right| \le |\psi|_3^3 \le A\|\psi\|_{\frac{1}{6}}^3 \le A\|\psi\|_0^{5/2}\|\psi\|_1^{1/2},$$

where A denotes various constants which are independent of ψ. Then Young's inequality gives

$$\left|\int \psi^3\ dx\right| \le \epsilon\|\psi\|_1^2 + A_\epsilon\|\psi\|_0^{10/3}$$

where $\epsilon > 0$ is arbitrary and A_ϵ depends on ϵ but not on ψ. Therefore, since $\|\psi\|_0 = |\psi|_2 = (Q(\psi))^{1/2}$, one has

$$\left|\int \psi^3\ dx\right| \le \epsilon\|\psi\|_1^2 + A_{\epsilon,q_1}$$

where again A_{ϵ,q_1} is independent of ψ. Hence

$$\begin{aligned}
E(\psi) &= E(\psi) + Q(\psi) - Q(\psi) \\
&= \frac{1}{2}\int \left[(\psi')^2 + \psi^2\right]\ dx - \frac{1}{6}\int \psi^3\ dx - Q(\psi) \\
&\geq \frac{1}{2}\|\psi\|_1^2 - \frac{\epsilon}{6}\|\psi\|_1^2 - \frac{1}{6}A_{\epsilon,q_1} - q_1.
\end{aligned}$$

Choosing $\epsilon = \epsilon_0 \leq 3$ then gives the lower bound

$$E(\psi) \geq -\frac{1}{6}A_{\epsilon_0,q_1} - q_1,$$

and so the proof is complete. □

LEMMA 2.3. *If $\{f_n\}$ is a minimizing sequence for I_q, then there exist constants $B > 0$ and $\delta > 0$ such that*

1. *$\|f_n\|_1 \leq B$ for all n and*
2. *$|f_n|_3 \geq \delta$ for all sufficiently large n.*

PROOF. To prove statement 1, write

$$\begin{aligned}
\frac{1}{2}\|f_n\|_1^2 &= E(f_n) + Q(f_n) + \frac{1}{6}\int f_n^3\ dx \\
&\leq \sup_n E(f_n) + q + \frac{A}{6}\|f_n\|_{1/6}^3 \\
&\leq A\left(1 + \|f_n\|_0^{5/2}\|f_n\|_1^{1/2}\right) \\
&\leq A\left(1 + \|f_n\|_1^{1/2}\right),
\end{aligned}$$

where again Sobolev embedding and interpolation theorems have been used, and A denotes various constants which are independent of n. Since the square of $\|f_n\|_1$ has now been bounded by a smaller power, the existence of the desired bound B follows.

To prove statement 2, we argue by contradiction: if no such constant δ exists, then

$$\liminf_{n\to\infty} \int (f_n)^3\ dx \leq 0,$$

so

$$I_q = \lim_{n\to\infty}\left(\frac{1}{2}\int (f_n')^2\ dx - \frac{1}{6}\int f_n^3\ dx\right) \geq \liminf_{n\to\infty}\left(-\frac{1}{6}\int f_n^3\ dx\right) \geq 0,$$

contradicting Lemma 2.2. □

LEMMA 2.4. *For all $q_1, q_2 > 0$, one has*

$$I_{(q_1+q_2)} < I_{q_1} + I_{q_2}.$$

PROOF. First we claim that for all $\theta > 0$ and $q > 0$,

$$I_{\theta q} = \theta^{5/3} I_q.$$

To see this, associate to each function $\psi \in H^1$ the function ψ_θ defined by

$$\psi_\theta(x) = \theta^{2/3}\psi(\theta^{1/3}x).$$

Then

$$Q(\psi_\theta) = \theta Q(\psi),$$

while

$$E(\psi_\theta) = \theta^{5/3} E(\psi).$$

Hence

$$\begin{aligned} I_{\theta q} &= \inf\{E(\psi_\theta) : Q(\psi_\theta) = \theta q\} \\ &= \inf\left\{\theta^{5/3} E(\psi) : Q(\psi) = q\right\} \\ &= \theta^{5/3} I_q, \end{aligned}$$

as claimed.

Now from the claim and Lemma 2.2 it follows that for all $q_1, q_2 > 0$

$$I_{(q_1+q_2)} = (q_1+q_2)^{5/3} I_1 < \left(q_1^{5/3} + q_2^{5/3}\right) I_1 = I_{q_1} + I_{q_2}.$$

$\square$

Next we consider separately the three possibilities $\alpha = q$, $0 < \alpha < q$, and $\alpha = 0$. The first of these is called the case of *compactness* by Lions because of the following lemma.

LEMMA 2.5 [**L1**]. *Suppose $\alpha = q$. Then there exists a sequence of real numbers $\{y_1, y_2, y_3, \dots\}$ such that*

1. *for every $z < q$ there exists $r = r(z)$ such that*

$$\int_{y_n - r}^{y_n + r} |f_n|^2 \, dx > z$$

for all sufficiently large n.

2. *the sequence $\{\tilde{f}_n\}$ defined by*

$$\tilde{f}_n(x) = f_n(x + y_n) \quad \textit{for } x \in \mathbf{R}$$

has a subsequence which converges in H^1 norm to a function $g \in G_q$. In particular, G_q is nonempty.

PROOF. Since $\alpha = q$, then there exists r_0 such that for all sufficiently large values of n we have

$$Q_n(r_0) = \sup_{y \in \mathbf{R}} \int_{y-r_0}^{y+r_0} |f_n|^2 \, dx > q/2.$$

Hence for each sufficiently large n we can find y_n such that

$$\int_{y_n - r_0}^{y_n + r_0} |f_n|^2 \, dx > q/2.$$

Now let $z < q$ be given; clearly we may assume $z > q/2$. Again, since $\alpha = q$ then we can find $r_0(z)$ and $N(z)$ such that if $n \geq N(z)$ then

$$\int_{y_n(z)-r_0(z)}^{y_n(z)+r_0(z)} |f_n|^2 \, dx > z$$

for some $y_n(z) \in \mathbf{R}$. Since $\int_{\mathbf{R}} |f_n|^2 \, dx = q$, it follows that for large n the intervals $[y_n - r_0, y_n + r_0]$ and $[y_n(z) - r_0(z), y_n(z) + r_0(z)]$ must overlap. Therefore, defining $r = r(z) = 2r_0(z) + r_0$, we have that $[y_n - r, y_n + r]$ contains $[y_n(z) - r_0(z), y_n(z) + r_0(z)]$, and statement 1 then follows.

Now statement 1 implies that for every $k \in \mathbf{N}$, there exists $r_k \in \mathbf{R}$ such that for all sufficiently large n,

$$\int_{-r_k}^{r_k} |\tilde{f}_n|^2 \, dx > 1 - \frac{1}{k}.$$

By Lemma 2.3.1, the sequence $\{\tilde{f}_n\}$ is uniformly bounded in H^1, and hence from the compactness of the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ on bounded intervals Ω it follows that some subsequence of $\{\tilde{f}_n\}$ converges in $L^2[-r_k, r_k]$ norm to a limit function $g \in L^2[-r_k, r_k]$ satisfying

$$\int_{-r_k}^{r_k} |g|^2 \, dx > 1 - \frac{1}{k}.$$

A Cantor diagonalization argument, together with the fact that $\int_{\mathbf{R}} |\tilde{f}_n|^2 \, dx = q$ for all n, then shows that some subsequence of $\{\tilde{f}_n\}$ converges in $L^2(\mathbf{R})$ norm to a function $g \in L^2(\mathbf{R})$ satisfying $\int_{\mathbf{R}} |g|^2 \, dx = q$. Again using Lemma 2.3.1, we have

$$|\tilde{f}_n - g|_3 \leq A\|\tilde{f}_n - g\|_{1/6} \leq A\|\tilde{f}_n - g\|_1^{1/6} \|\tilde{f}_n - g\|_0^{5/6} \leq A|\tilde{f}_n - g|_2^{5/6},$$

where A is independent of n; so $\tilde{f}_n \to g$ in L^3 norm also. Furthermore, by the weak compactness of the unit sphere and the weak lower semicontinuity of the norm in Hilbert space, we know that $\tilde{f}_n$ converges weakly to g in H^1, and that

$$\|g\|_1 \leq \liminf_{n\to\infty} \|\tilde{f}_n\|_1.$$

It follows that

$$E(g) \leq \lim_{n\to\infty} E(\tilde{f}_n) = I_q,$$

whence $E(g) = I_q$ and $g \in G_q$. Finally, $E(g) = \lim_{n\to\infty} E(\tilde{f}_n)$, $|g|_3 = \lim_{n\to\infty} |\tilde{f}_n|_3$, and $|g|_2 = \lim_{n\to\infty} |\tilde{f}_n|_2$ together imply that $\|g\|_1 = \lim_{n\to\infty} \|\tilde{f}_n\|_1$, and from an elementary exercise in Hilbert space theory it then follows that $\tilde{f}_n$ converges to g in H^1 norm. $\square$

The next lemma is used to describe the behavior of minimizing sequences in the case $0 < \alpha < q$.

LEMMA 2.6. *For every $\epsilon > 0$, there exist a number $N \in \mathbf{N}$ and sequences $\{g_N, g_{N+1}, \dots\}$ and $\{h_N, h_{N+1}, \dots\}$ of H^1 functions such that for every $n \geq N$,*

1. $|Q(g_n) - \alpha| < \epsilon$
2. $|Q(h_n) - (q - \alpha)| < \epsilon$
3. $E(f_n) \geq E(g_n) + E(h_n) - \epsilon$

Proof. Choose $\phi \in C_0^\infty[-2,2]$ such that $\phi \equiv 1$ on $[-1,1]$, and let $\psi \in C^\infty(\mathbf{R})$ be such that $\phi^2 + \psi^2 \equiv 1$ on $\mathbf{R}$. For each $r \in \mathbf{R}$, define $\phi_r(x) = \phi(x/r)$ and $\psi_r(x) = \psi(x/r)$.

For all sufficiently large values of r we have

$$\alpha - \epsilon < M(r) \leq M(2r) \leq \alpha.$$

Assume for the moment that such a value of r has been chosen. Then we can choose N so large that

$$\alpha - \epsilon < M_n(r) \leq M_n(2r) < \alpha + \epsilon$$

for all $n \geq N$. Hence for each $n \geq N$ we can find y_n such that

$$\int_{y_n - r}^{y_n + r} |f_n|^2 \, dx > \alpha - \epsilon \tag{2.1}$$

and

$$\int_{y_n - 2r}^{y_n + 2r} |f_n|^2 \, dx < \alpha + \epsilon. \tag{2.2}$$

Define $g_n(x) = \phi_r(x - y_n) f_n(x)$ and $h_n(x) = \psi_r(x - y_n) f_n(x)$. Then clearly statements 1 and 2 are satisfied by g_n and h_n.

To prove statement 3, note that

$$\begin{aligned} &E(g_n) + E(h_n) = \\ &= \frac{1}{2}\left[\int \phi_r^2 (f_n')^2 \, dx + 2\int \phi_r \phi_r' f_n f_n' \, dx + \int (\phi_r')^2 f_n^2 \, dx\right] \\ &\quad + \frac{1}{2}\left[\int \psi_r^2 (f_n')^2 \, dx + 2\int \psi_r \psi_r' f_n f_n' \, dx + \int (\psi_r')^2 f_n^2 \, dx\right] \\ &\quad - \frac{1}{6}\int \phi_r^2 f_n^3 \, dx - \frac{1}{6}\int \psi_r^2 f_n^3 \, dx \\ &\quad + \frac{1}{6}\int (\phi_r^2 - \phi_r^3) f_n^3 \, dx + \frac{1}{6}\int (\psi_r^2 - \psi_r^3) f_n^3 \, dx, \end{aligned}$$

where for brevity we have written simply ϕ_r and ψ_r for the functions $\phi_r(x - y_n)$ and $\psi_r(x - y_n)$. Now $\phi_r^2 + \psi_r^2 \equiv 1$, $|(\phi_r)'|_\infty = |\phi'|_\infty / r$, and $|(\psi_r)'|_\infty = |\psi'|_\infty / r$. Therefore, making use of Hölder's Inequality and Lemma 2.3.1, one can rewrite the preceding equation in the form

$$E(g_n) + E(h_n) = E(f_n) + O(1/r) + \frac{1}{6}\int \left[(\phi_r^2 - \phi_r^3) + (\psi_r^2 - \psi_r^3)\right] f_n^3 \, dx,$$

where $O(1/r)$ signifies a term bounded in absolute value by A_1/r with A_1 independent of r and n. But using (2.1) and (2.2) we obtain

$$\left|\int \left[(\phi_r^2 - \phi_r^3) + (\psi_r^2 - \psi_r^3)\right] f_n^3 \, dx\right| \leq \left(\int_{r \leq |x - y_n| \leq 2r} 2|f_n|^2 \, dx\right) \cdot |f|_\infty \leq A_2 \epsilon,$$

where again A_2 is independent of r and n.

It is now time to choose r, and we make the choice so large that the $O(1/r)$ term in the preceding paragraph is less than ϵ in absolute value. For the corresponding choices of sequences $\{g_n\}$ and $\{h_n\}$, statements 1 and 2 hold together with

$$E(f_n) \geq E(g_n) + E(h_n) - (A_2 + 1)\epsilon \tag{2.3}$$

for all $n \geq N(r)$. Finally, we may return to the beginning of the proof and there replace ϵ by $\min(\epsilon, \epsilon/(A_2+1))$, thus transforming (2.3) into statement 3 (without affecting statements 1 and 2). □

Corollary 2.7. *If $0 < \alpha < q$ then*

$$I_q \geq I_\alpha + I_{q-\alpha}.$$

Proof. First observe that if g is a function such that $|Q(g) - \alpha| < \epsilon$, then $Q(\beta g) = \alpha$, where $\beta = \sqrt{\alpha/Q(g)}$ satisfies $|\beta - 1| < A_1\epsilon$ with A_1 independent of g and ϵ. Hence

$$I_\alpha \leq E(\beta g) \leq E(g) + A_2\epsilon,$$

where A_2 depends only on A_1 and $\|g\|_1$. A similar result holds for functions h such that $|Q(h) - (q-\alpha)| < \epsilon$.

From these observations and Lemma 2.6 it follows easily that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and corresponding functions g_{n_k} and h_{n_k} such that for all k,

$$\begin{aligned} E(g_{n_k}) &\geq I_\alpha - \frac{1}{k}, \\ E(h_{n_k}) &\geq I_{q-\alpha} - \frac{1}{k}, \quad \text{and} \\ E(f_{n_k}) &\geq E(g_{n_k}) + E(h_{n_k}) - \frac{1}{k}. \end{aligned}$$

Hence

$$E(f_{n_k}) \geq I_\alpha + I_{q-\alpha} - \frac{3}{k}.$$

The desired result is now obtained by taking the limit of both sides as $k \to \infty$. □

Remark. Because of Lemma 2.6 and its corollary, Lions calls $0 < \alpha < q$ the case of *dichotomy*: each of the minimizing functions f_n can be split into two summands which carry fixed proportions of the constraint functional Q, and which are sufficiently separated spatially that the sum of the values of E at each summand does not exceed $E(f_n)$. In fact, as explained in the remark following Theorem 2.9 below, for a general class of variational problems of the type considered here, the inequality $I_q \leq I_\beta + I_{q-\beta}$ holds for all $\beta \in (0, q)$, so that in the case of dichotomy one would have from Corollary 2.7 that

$$I_q = I_\alpha + I_{q-\alpha}.$$

Thus the two sequences of summands will themselves be minimizing sequences for I_α and $I_{q-\alpha}$ respectively.

Our final lemma shows that the possibility $\alpha = 0$ (called the case of *vanishing* by Lions) does not occur here.

Lemma 2.8. *For every minimizing sequence, $\alpha > 0$.*

Proof. From Lemmas 2.1 and 2.3 we conclude that there exists $\eta > 0$ and a sequence $\{y_n\}$ of real numbers such that

$$\int_{y_n-1/2}^{y_n+1/2} |f_n|^3 \, dx \geq \eta$$

for all n. Hence

$$\eta \leq |f_n|_\infty \left(\int_{y_n-1/2}^{y_n+1/2} |f_n|^2 \, dx \right) \leq AB \left(\int_{y_n-1/2}^{y_n+1/2} |f_n|^2 \, dx \right),$$

where A is the Sobolev constant in the embedding of L^∞ into H^1. It follows that $\alpha = \lim_{r\to\infty} M(r) \geq M(1/2) = \lim_{n\to\infty} M_n(1/2) \geq \dfrac{\eta}{AB} > 0$. □

Theorem 2.9. *The set G_q is not empty. Moreover, if $\{f_n\}$ is any minimizing sequence for I_q, then*

1. *there exists a sequence $\{y_1, y_2, \dots\}$ and an element $g \in G_q$ such that $f_n(\cdot + y_n)$ has a subsequence converging strongly in H^1 to g.*
2. $$\lim_{n\to\infty} \inf_{\substack{g\in G_q \\ y\in\mathbf{R}}} \|f_n(\cdot + y) - g\|_1 = 0.$$
3. $$\lim_{n\to\infty} \inf_{g\in G_q} \|f_n - g\|_1 = 0.$$

Proof. From Lemmas 2.4, 2.7 and 2.8 it follows that $\alpha = q$. Hence by Lemma 2.5 the set G_q is nonempty and statement 1 of the present lemma holds.

Now suppose that statement 2 does not hold; then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a number $\epsilon > 0$ such that

$$\inf_{\substack{g\in G_q \\ y\in\mathbf{R}}} \|f_{n_k}(\cdot + y) - g\|_1 \geq \epsilon$$

for all $k \in \mathbf{N}$. But since $\{f_{n_k}\}$ is itself a minimizing sequence for I_q, from statement 1 it follows that there exist a sequence $\{y_k\}$ and $g_0 \in G_q$ such that

$$\liminf_{k\to\infty} \|f_{n_k}(\cdot + y_k) - g_0\|_1 = 0.$$

This contradiction proves statement 2.

Finally, since the functionals E and Q are invariant under translations, then G_q clearly contains any translate of g if it contains g, and hence statement 3 follows immediately from statement 2. □

Remark. For arbitrary functionals E and Q defined on a function space X, and I_q and G_q defined as above, the preceding arguments show that under general conditions the subadditivity property $I_q < I_\beta + I_{q-\beta}$ (for all $\beta \in (0, q)$) is sufficient to imply the relative compactness (modulo translations) of all minimizing sequences for I_q. Lions [**L1**] has also given a heuristic argument, which we now paraphrase here, for the necessity of this property. Let $\beta \in (0, q)$ be given, and let $\{g_n\}$ and $\{h_n\}$ be minimizing sequences for I_β, $I_{q-\beta}$ respectively. Define $\tilde{h}_n = h_n(\cdot + y_n)$

where y_n is chosen so that the distance between the supports of g_n and $\tilde{h}_n$ tends to infinity with n. If E and Q are local operators, or at least not too nonlocal, then

$$E(g_n + \tilde{h}_n) \sim E(g_n) + E(\tilde{h}_n) \to I_\beta + I_{q-\beta}$$

and

$$Q(g_n + \tilde{h}_n) \sim Q(g_n) + Q(\tilde{h}_n) \to q,$$

as $n \to \infty$, so that

$$I_\beta + I_{q-\beta} = \lim_{n\to\infty} E(g_n + \tilde{h}_n) \geq I_q.$$

Now if $I_q = I_\beta + I_{q-\beta}$ then $g_n + \tilde{h}_n$ is a minimizing sequence for I_q; but $g_n + \tilde{h}_n$ cannot have a strongly convergent subsequence (even after translations) in any of the usual function spaces. (For example, it is an easy exercise to show that if $\int g_n^2 = \beta > 0$ and $\int \tilde{h}_n^2 = q - \beta > 0$ for all n, and the distance between the supports of g_n and $\tilde{h}_n$ tends to infinity with n, then there does not exist any sequence $\{y_n\}$ of real numbers such that $\{g_n(\cdot + y_n) + \tilde{h}_n(\cdot + y_n)\}$ has a convergent subsequence in L^2.) Hence if all minimizing sequences for I_q are relatively compact (modulo translations), then we must have $I_q < I_\beta + I_{q-\beta}$ for all $\beta \in (0, q)$.

An immediate consequence of Theorem 2.9 is that G_q forms a stable set for the initial-value problem for (1.1).

Corollary 2.10. *For every $\epsilon > 0$, there exists $\delta > 0$ such that if*

$$\inf_{g \in G_q} \|u_0 - g\|_1 < \delta,$$

then the solution $u(x,t)$ of (1.1) with $u(x,0) = u_0$ satisfies

$$\inf_{g \in G_q} \|u(\cdot, t) - g\|_1 < \epsilon$$

for all $t \in \mathbf{R}$.

Proof. Suppose the theorem to be false; then there exist a number $\epsilon > 0$, a sequence $\{\psi_n\}$ of functions in H^1, and a sequence of times $\{t_n\}$ such that

$$\inf_{g \in G_q} \|\psi_n - g\|_1 < \frac{1}{n}$$

and

$$\inf_{g \in G_q} \|u_n(\cdot, t_n) - g\|_1 \geq \epsilon$$

for all n, where $u_n(x,t)$ solves (1.1) with $u_n(x,0) = \psi_n$. Then since $\psi_n \to G_q$ in H^1, and $E(g) = I_q$ and $Q(g) = q$ for $g \in G_q$, we have $E(\psi_n) \to I_q$ and $Q(\psi_n) \to q$. Choose $\{\alpha_n\}$ such that $Q(\alpha_n \psi_n) = q$ for all n; thus $\alpha_n \to 1$. Hence the sequence $f_n = \alpha_n u_n(\cdot, t_n)$ satisfies $Q(f_n) = q$ and

$$\lim_{n\to\infty} E(f_n) = \lim_{n\to\infty} E(u_n(\cdot, t_n)) = \lim_{n\to\infty} E(\psi_n) = I_q,$$

and is therefore a minimizing sequence for I_q. From Theorem 2.9.3 it follows that for all n sufficiently large there exists $g_n \in G_q$ such that $\|f_n - g_n\|_1 < \frac{\epsilon}{2}$. But then

$$\epsilon \leq \|u_n(\cdot, t_n) - g_n\|_1 \leq \|u_n(\cdot, t_n) - f_n\|_1 + \|f_n - g_n\|_1 \leq |1 - \alpha_n| \cdot \|u_n(\cdot, t_n)\|_1 + \frac{\epsilon}{2},$$

and taking $n \to \infty$ gives $\epsilon \leq \frac{\epsilon}{2}$, a contradiction. $\square$

Note that the stability result in Corollary 2.10 is weaker than that of Theorem 1.1; this is an instance of the general fact, discussed above in Section 1, that the concentration-compactness method by itself only proves stability with respect to an unspecified set of minimizers. If for example the set G_q contained two functions g_1 or g_2 which were not translates of each other, then not all minimizing sequences for I_q would converge modulo translations; since a minimizing sequence could contain two subsequences tending to g_1 and g_2 respectively. Of course if the elements of G_q were isolated points (modulo translations) in function space, then the argument used to prove Corollary 2.10 would show that each such point constitutes a stable set in itself; but nonisolated minimizers contained in G_q could fail to be individually stable.

In the present case, however, it is easy to see that G_q contains but a single function (modulo translations), and that this function is indeed a solitary-wave solution of (1.1).

Proposition 2.11. *If G_q is not empty then*

$$G_q = \{\phi_C(\cdot + x_0) : x_0 \in \mathbf{R}\}\,.$$

Proof. If $g(x) \in G_q$ then by the Lagrange multiplier principle (see, e.g., Theorem 7.7.2 of [**Lu**]), there exists $\lambda \in \mathbf{R}$ such that

$$\delta E(g) + \lambda\ \delta Q(g) = 0, \tag{2.4}$$

where $\delta E(g)$ and $\delta Q(g)$ are the Fréchet derivatives of E and Q at g. Now δE and δQ are given (as distributions in H^{-1}) by

$$\begin{aligned} \delta E(g) &= -g'' - \frac{1}{2}g^2, \\ \delta Q(g) &= g; \end{aligned}$$

and therefore (2.4) is an ordinary differential equation in g. A bootstrap argument shows that any L^2 distribution solution of (2.4) must be smooth, and from an elementary phase plane analysis one then concludes that the only solutions of (2.4) in L^2 are the functions $\phi_\lambda(x + x_0)$, where x_0 is arbitrary and ϕ_λ is defined by (1.2) with C replaced by λ. But it is easily seen that $Q(\phi_\lambda) = q = Q(\phi_C)$ if and only if $\lambda = C$. Hence $g(x) = \phi_C(x + x_0)$. This proves that $G_q \subset \{\phi_C(\cdot + x_0) : x_0 \in \mathbf{R}\}$, and the reverse inclusion follows (if G_q is not empty) from the translation invariance of E and Q. □

Combining Theorem 2.9 and its corollary with Proposition 2.11, we see that Theorem 1.1 has now been completely proved.

3. An application to a nonlocal equation

In this section we illustrate the use of the concentration-compactness method for nonlocal equations by proving the stability of solitary-wave solutions of the equation

$$u_t + uu_x - (Lu)_x = 0, \tag{3.1}$$

derived by Kubota, Ko and Dobbs [**KKD**] as a model for long internal waves in a stratified fluid. Here L is the Fourier multiplier operator defined by

$$\widehat{Lw}(k) = m(k)\widehat{w}(k),$$

where

$$m(k) = \beta_1 \left(k \coth(kH_1) - \frac{1}{H_1} \right) + \beta_2 \left(k \coth(kH_2) - \frac{1}{H_2} \right)$$

and β_1, β_2, H_1, H_2 are positive constants. By a *solitary-wave solution* of (3.1) we mean a solution of the form $u_S(x,t) = g(x - Ct)$, where $C \in \mathbf{R}$; but often we will abuse terminology slightly and use the term solitary wave to refer to the profile function g corresponding to such a solution. Here "solution" means "classical solution"; it is no gain of generality to consider distribution solutions, since it turns out that if $g \in L^2$ and u_S satisfies (3.1) in the sense of distributions, then g must in fact be infinitely differentiable (see Lemma 3.3 of [**ABS**]).

When $H_1 = H_2$ (or when one of β_1 or β_2 is zero), equation (3.1) has the structure of a completely integrable Hamiltonian system [**KSA,LR,R**] and explicit multisoliton solutions are known [**J,JE**]. In particular, the solitary-wave solutions of (3.1) in this case are given by $u(x,t) = \phi_{C,H}(x + x_0 - Ct)$, where $C > 0$ and $x_0 \in \mathbf{R}$ are arbitrary, H denotes the common value of H_1 and H_2, and

$$\phi_{C,H}(\xi) = \left[\frac{2a(\beta_1 + \beta_2) \sin aH}{\cosh a\xi + \cos aH} \right],$$

with $a \in (0, \pi/H)$ determined by the equation

$$aH \cot aH = 1 - \frac{CH}{\beta_1 + \beta_2}.$$

The stability of these solitary waves was studied in [**AB**], where it was shown that a result similar to Theorem 1.1 holds for $\phi_{C,H}$ for all positive values of C and H.

In the general case when H_1 is not equal to H_2, equation (3.1) does not appear to be completely integrable (cf. [**BD**]), nor is there any known explicit formula for solitary waves. However, in [**ABS**] it is shown that solitary-wave solutions do exist for all positive values of β_1, β_2, H_1 and H_2 and all positive wavespeeds C. Moreover, when H_1 is near H_2 one has the following stability result:

THEOREM 3.1 [**ABS**]. *Let β_1, β_2, H, C be arbitrary positive numbers. Then there exists $\eta > 0$ such that if $H_1 = H$ and $|H_2 - H_1| < \eta$ then (3.1) has a solution $u_S(x,t) = \phi(x - Ct)$ which is stable in the following sense: for every $\epsilon > 0$, there exists $\delta > 0$ such that if*

$$\inf_{y \in \mathbf{R}} \|u_0 - \phi(\cdot + y)\|_{\frac{1}{2}} < \delta,$$

then the solution $u(x,t)$ of (3.1) with $u(x,0) = u_0$ satisfies

$$\inf_{y \in \mathbf{R}} \|u(\cdot,t) - \phi(\cdot + y)\|_{\frac{1}{2}} < \epsilon$$

for all $t \in \mathbf{R}$.

This result leaves open the question of whether stable solitary-wave solutions of (3.1) exist when H_1 is not close to H_2. The following theorem answers this question in the affirmative, albeit with possibly a broader interpretation of stability than that given in Theorem 3.1.

THEOREM 3.2. *Let β_1, β_2, H_1, H_2 be arbitrary positive numbers. For each $q > 0$ there exists a nonempty set G_q, consisting of solitary-wave solutions g of (3.1) having positive wavespeeds and satisfying $Q(g) = q$, which is stable in the following sense: for every $\epsilon > 0$ there exists $\delta > 0$ such that if*

$$\inf_{g \in G_q} \|u_0 - g\|_{\frac{1}{2}} < \delta,$$

then the solution $u(x,t)$ of (3.1) with $u(x,0) = u_0$ satisfies

$$\inf_{g \in G_q} \|u(\cdot,t) - g\|_{\frac{1}{2}} < \epsilon$$

for all $t \in \mathbf{R}$.

We will prove Theorem 3.2 following the same steps as used in the preceding section to prove Theorem 2.9 and its corollary. First define functionals Q and E on $H^{1/2}$ by

$$Q(u) = \frac{1}{2}\int_{-\infty}^{\infty} u^2 \, dx$$

and

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty} \left[uLu - \frac{1}{3}u^3 \right] \, dx,$$

and give I_q and G_q the same definitions as in Section 2. To each minimizing sequence for I_q we associate a number $\alpha \in [0, q]$ using the same procedure as in Section 2.

Analogues of Lemmas 2.1 through 2.4 are as follows.

LEMMA 3.3. *Suppose $B > 0$ and $\delta > 0$ are given. Then there exists $\eta = \eta(B, \delta) > 0$ such that if $f \in H^{1/2}$ with $\|f\|_{1/2} \le B$ and $|f|_3 \ge \delta$, then*

$$\sup_{y \in \mathbf{R}} \int_{y-2}^{y+2} |f(x)|^3 \, dx \ge \eta.$$

PROOF. The proof of this lemma is contained in the proofs of Lemmas 3.7, 3.8, and 3.9 of [**ABS**]; but for completeness we recall the proof here. Choose a smooth function $\zeta : \mathbf{R} \to [0,1]$ with support in $[-2,2]$ and satisfying $\sum_{j \in \mathbf{Z}} \zeta(x-j) = 1$ for all $x \in \mathbf{R}$; and define $\zeta_j(x) = \zeta(x-j)$ for $j \in \mathbf{Z}$. The map $T : H^s \to l_2(H^s)$ defined by

$$Tf = \{\zeta_j f\}_{j \in \mathbf{Z}}$$

is clearly bounded for $s = 0$ and $s = 1$, and hence by interpolation ([**BL**], Section 5.6) is also bounded for $s = 1/2$; that is, there exists $A_0 > 0$ such that for all $f \in H^{1/2}$,

$$\sum_{j \in \mathbf{Z}} \|\zeta_j f\|_{\frac{1}{2}}^2 \le A_0 \|f\|_{\frac{1}{2}}^2.$$

Now let A_1 be a positive number such that $\sum_{j \in \mathbf{Z}} |\zeta(x-j)|^3 \ge A_1$ for all $x \in \mathbf{R}$. We claim that for every function $f \in H^{1/2}$ which is not identically zero, there exists an integer j_0 such that

$$\|\zeta_{j_0} f\|_{\frac{1}{2}}^2 \le \left(1 + A_2 |f|_3^{-3}\right) |\zeta_{j_0} f|_3^3,$$

where $A_2 = A_0 B^2 / A_1$. To see this, assume to the contrary that

$$\|\zeta_j f\|_{\frac{1}{2}}^2 > \left(1 + A_2 |f|_3^{-3}\right) |\zeta_j f|_3^3$$

holds for every $j \in \mathbf{Z}$. After summing over j we obtain

$$A_0\|f\|_{\frac{1}{2}}^2 > \left(1 + A_2|f|_3^{-3}\right)\sum_{j\in\mathbf{Z}} |\zeta_j f|_3^3,$$

and hence

$$A_0B^2 > \left(1 + A_2|f|_3^{-3}\right) A_1|f|_3^3 = A_1|f|_3^3 + A_0B^2,$$

which is a contradiction.

Finally, observe that from the claim just proved and the assumptions of the lemma it follows that

$$\|\zeta_{j_0} f\|_{\frac{1}{2}}^2 \le \left(1 + A_2/\delta^3\right)|\zeta_{j_0} f|_3^3,$$

whereas by the Sobolev embedding theorem one has

$$|\zeta_{j_0} f|_3 \le A_3\|\zeta_{j_0} f\|_{\frac{1}{2}},$$

with a constant A_3 that is independent of f. Hence

$$|\zeta_{j_0} f|_3 \ge \left[A_3^2(1 + A_2/\delta^3)\right]^{-1},$$

and since

$$\int_{j_0-2}^{j_0+2} |f_j|^3\ dx \ge |\zeta_{j_0} f|_3^3,$$

the statement of the lemma follows immediately with $\eta = \left[A_3^2(1 + A_2/\delta^3)\right]^{-3}$. $\square$

Lemma 3.4. *For all $q_1 > 0$, one has*

$$-\infty < I_{q_1} < 0.$$

Proof. Choose $\psi \in H^{1/2}$ such that $Q(\psi) = q_1$ and $\int \psi^3\ dx \neq 0$, and for each $\theta > 0$, define ψ_θ by $\psi_\theta(x) = \sqrt{\theta}\psi(\theta x)$. Observe that $0 \le m(k) \le (\beta_1 + \beta_2)|k|$ for all $k \in \mathbf{R}$. Hence, using Parseval's identity and taking into account the action of dilation on Fourier transforms, we have

$$\begin{aligned}\int \psi_\theta L(\psi_\theta)\ dx &= \frac{1}{\theta}\int m(k)|\widehat{\psi}(k/\theta)|^2\ dk \\ &= \int m(\theta k)|\widehat{\psi}(k)|^2\ dk \le (\beta_1+\beta_2)\theta\|\psi\|_{\frac{1}{2}}^2.\end{aligned}$$

Therefore

$$E(\psi_\theta) \le \frac{(\beta_1+\beta_2)\theta}{2}\ \|\psi\|_{\frac{1}{2}}^2 - \frac{\theta^{1/2}}{6}\int \psi^3\ dx,$$

and so for $\theta = \theta_0$ sufficiently small one has $E(\psi_{\theta_0}) < 0$. Since $Q(\psi_{\theta_0}) = q_1$ it follows that $I_{q_1} < 0$.

To show that $I_{q_1} > -\infty$, we proceed as in the proof of Lemma 2.2, except that here we use the estimates

$$\left|\int \psi^3\ dx\right| \le A\|\psi\|_{\frac{1}{6}}^3 \le A\|\psi\|_0^2\|\psi\|_{\frac{1}{2}} \le \epsilon\|\psi\|_{\frac{1}{2}}^2 + A_\epsilon\|\psi\|_0^4,$$

valid for any $\epsilon > 0$ with A_ϵ depending only on ϵ, and

$$\begin{aligned} E(\psi) &= E(\psi) + Q(\psi) - Q(\psi) \\ &= \frac{1}{2}\int \left[(\psi L\psi + \psi^2\right] \, dx - \frac{1}{6}\int \psi^3 \, dx - Q(\psi) \\ &\geq \frac{A}{2}\|\psi\|_{\frac{1}{2}}^2 - \frac{\epsilon}{6}\|\psi\|_1^2 - \frac{1}{6}A_{\epsilon,q_1} - q_1, \end{aligned}$$

where A_{ϵ,q_1} depends only on ϵ and q_1, and A is chosen so that $1 + m(k) \geq A|k|$ for all $k \in \mathbf{R}$. □

Lemma 3.5. *If $\{f_n\}$ is a minimizing sequence for I_q, then there exist constants $B > 0$ and $\delta > 0$ such that*

1. *$\|f_n\|_{\frac{1}{2}} \leq B$ for all n and*
2. *$|f_n|_3 \geq \delta$ for all sufficiently large n.*

Proof. Choosing A such that $1 + m(k) \geq A|k|$ for all $k \in \mathbf{R}$, we have by Parseval's inequality and Sobolev embedding and interpolation theorems that

$$\begin{aligned} \frac{A}{2}\|f_n\|_{\frac{1}{2}}^2 &\leq E(f_n) + Q(f_n) + \frac{1}{6}\int f_n^3 \, dx \\ &\leq \sup_n E(f_n) + q + \frac{A}{6}\|f_n\|_{1/6}^3 \\ &\leq A\left(1 + \|f_n\|_0^2\|f_n\|_{\frac{1}{2}}\right) \\ &\leq A\left(1 + \|f_n\|_{\frac{1}{2}}\right). \end{aligned}$$

From here the proof is the same as the proof of Lemma 2.3. □

Lemma 3.6. *For all $q_1, q_2 > 0$, one has*

$$I_{(q_1+q_2)} < I_{q_1} + I_{q_2}.$$

Proof. Since $m(k)$ is not a homogeneous function of k, we cannot use the argument in the proof of Lemma 2.4. Instead, we use an argument from [**L2**], pp. 228-229. First we claim that for $\theta > 1$ and $q > 0$,

$$I_{\theta q} < \theta I_q.$$

To see this, let $\{f_n\}$ be a minimizing sequence for I_q, and define $\tilde{f}_n = \sqrt{\theta} f_n$ for all n, so that $Q(\tilde{f}_n) = \theta q$ and hence $E(\tilde{f}_n) \geq I_{\theta q}$ for all n. Then for all n we have

$$I_{\theta q} \leq E(\tilde{f}_n) = \frac{1}{2}\int \left[\tilde{f}_n L\tilde{f}_n - \frac{1}{3}\tilde{f}_n^3\right] \, dx = \theta E(f_n) + \frac{1}{6}(\theta - \theta^{3/2})\int f_n^3 \, dx.$$

Now taking $n \to \infty$ and using Lemma 3.5.2, we obtain

$$I_{\theta q} \leq \theta I_q + \frac{1}{6}(\theta - \theta^{3/2})\delta < \theta I_q,$$

and so the claim is proved.

Now suppose one of q_1 and q_2 is greater than the other, say $q_1 > q_2$. Then from the claim just proved, it follows that

$$\begin{aligned} I_{(q_1+q_2)} &= I_{q_1(1+q_2/q_1)} < (1+\frac{q_2}{q_1})I_{q_1} \\ &< I_{q_1} + \frac{q_2}{q_1}\left(\frac{q_1}{q_2}I_{q_2}\right) = I_{q_1} + I_{q_2}, \end{aligned}$$

as desired. Also, in the remaining case when $q_1 = q_2$, we have

$$I_{(q_1+q_2)} = I_{2q_1} < 2I_{q_1} = I_{q_1} + I_{q_2},$$

and so the proof is complete. $\square$

The statement and proof of Lemma 2.5 go through unchanged in the present situation (except that H^1 is replaced by $H^{1/2}$) and so will not be repeated here. Before proceeding to the analogue of Lemma 2.6, we need the following result (cf. Lemma 3.10 of [**ABS**]).

LEMMA 3.7. *There exists a constant $A > 0$ such that if θ is any continuously differentiable function with θ and θ' in L^∞, and f is any L^2 function, then*

$$|[L,\theta]f|_2 \le A|\theta'|_\infty|f|_2,$$

where $[L,\theta]f$ denotes the commutator $L(\theta f) - \theta(Lf)$.

PROOF. By a standard density argument, it suffices to prove the result for arbitrary functions θ and f in $C_0^\infty(\mathbf{R})$.

Write $L = \frac{d}{dx}T$, where T is the Fourier multiplier operator defined by $\widehat{Tf}(k) = \sigma(k)\widehat{f}(k)$ with $\sigma(k) = m(k)/ik$. Since $\sigma(k)$ is bounded on $\mathbf{R}$, then T is a bounded operator on L_2. Moreover, it is easily verified that

$$\sup_{k\in\mathbf{R}} |k|^n \left|\left(\frac{d}{dk}\right)^n \sigma(k)\right| < \infty$$

for all $n \in \mathbf{N}$, and hence by Theorem 35 of [**CM**] there exists $A_1 > 0$ such that

$$|[T,\theta](f')|_2 \le A_1|\theta'|_\infty|f|_2$$

for all functions θ and f in $C_0^\infty(\mathbf{R})$. Therefore

$$\begin{aligned} |[L,\theta]f|_2 &= \left|T\frac{d}{dx}(\theta f) - \theta T\left(\frac{df}{dx}\right)\right|_2 \\ &\le |T(\theta' f)|_2 + |[T,\theta](f')|_2 \\ &\le \|T\|\ |\theta'|_\infty|f|_2 + A_1|\theta'|_\infty|f|_2, \end{aligned}$$

where $\|T\|$ denotes the norm of T as an operator on L_2. Thus the lemma has been proved with $A = \|T\| + A_1$. $\square$

LEMMA 3.8. *For every $\epsilon > 0$, there exist a number $N \in \mathbf{N}$ and sequences $\{g_N, g_{N+1}, \dots\}$ and $\{h_N, h_{N+1}, \dots\}$ of functions in $H^{1/2}$ such that for every $n \ge N$,*

1. $|Q(g_n) - \alpha| < \epsilon$
2. $|Q(h_n) - (q-\alpha)| < \epsilon$
3. $E(f_n) \ge E(g_n) + E(h_n) - \epsilon$

PROOF. As in the proof of Lemma 2.6, we choose $r \in \mathbf{R}$ and $N \in \mathbf{N}$ so large that

$$\alpha - \epsilon < M_n(r) \leq M_n(2r) < \alpha + \epsilon$$

for all $n \geq N$, and define sequences $\{g_n\}$ and $\{h_n\}$ by $g_n(x) = \phi_r(x - y_n)f_n(x)$ and $h_n(x) = \psi_r(x - y_n)f_n(x)$ where ϕ and ψ are as before and y_n is chosen so that (2.1) and (2.2) hold. Then statements 1 and 2 follow, and it remains to prove the third statement. We write

$$\begin{aligned} E(g_n) + E(h_n) = & \\ = \ & \frac{1}{2}\left[\int \phi_r^2 f_n L f_n \ dx + \int \phi_r f_n [L, \phi_r] f_n \ dx\right] \\ & + \frac{1}{2}\left[\int \psi_r^2 f_n L f_n \ dx + \int \psi_r f_n [L, \psi_r] f_n \ dx\right] \\ & - \frac{1}{6}\int \phi_r^2 f_n^3 \ dx - \frac{1}{6}\int \psi_r^2 f_n^3 \ dx \\ & + \frac{1}{6}\int (\phi_r^2 - \phi_r^3) f_n^3 \ dx + \frac{1}{6}\int (\psi_r^2 - \psi_r^3) f_n^3 \ dx, \end{aligned}$$

where again for brevity the arguments of the functions $\phi_r(x - y_n)$ and $\psi_r(x - y_n)$ have been omitted. Now using Hölder's inequality and Lemmas 3.5.1 and 3.7, and arguing as in the proof of Lemma 2.6, we obtain

$$E(g_n) + E(h_n) = E(f_n) + O(1/r) + O(\epsilon),$$

where $O(1/r)$ and $O(\epsilon)$ denote terms bounded by A/r and $A\epsilon$, with constants A independent of r and n. The proof now concludes as before. □

Corollary 2.7 holds in the present context without change of statement or proof, and the same is true of Lemma 2.8 except that here the last display in its proof should be modified to read

$$\begin{aligned} \eta &\leq \left(\int_{y_n-2}^{y_n+2} |f_n|^2 \ dx\right)^{1/2} \left(\int_{y_n-2}^{y_n+2} |f_n|^4 \ dx\right)^{1/2} \\ &\leq \left(\int_{y_n-2}^{y_n+2} |f_n|^2 \ dx\right)^{1/2} \left(\int_{-\infty}^{\infty} |f_n|^4 \ dx\right)^{1/2} \\ &\leq AB^2 \left(\int_{y_n-2}^{y_n+2} |f_n|^2 \ dx\right)^{1/2}, \end{aligned}$$

where A is the Sobolev constant in the embedding of L^4 into $H^{1/2}$.

Thus all the preliminaries for the proofs of Theorem 2.9 and Corollary 2.10 have been established, and these proofs now apply in the present context without modification (except for the replacement of the H^1 norm by the $H^{1/2}$ norm). To complete the demonstration of Theorem 3.2 it remains only to justify the statement made there that the set G_q consists of solitary-wave solutions of (3.1) having positive wavespeeds. If $g \in G_q$ then by the Lagrange multiplier principle (cf. the proof of Proposition 2.11) there exists $\lambda \in \mathbf{R}$ such that

$$\delta E(g) + \lambda \ \delta Q(g) = 0,$$

where the Fréchet derivatives δE and δQ are given by

$$\begin{aligned} \delta E(g) &= Lg - \frac{1}{2}g^2, \\ \delta Q(g) &= g. \end{aligned}$$

Hence $u(x,t) = g(x-\lambda t)$ solves (3.1), or in other words g is a solitary-wave solution of (3.1) with wavespeed λ. To see that $\lambda > 0$, note first that

$$\begin{aligned} \frac{d}{d\theta}\left[E(\theta g)\right]_{\theta=1} &= \frac{d}{d\theta}\left[\frac{\theta^2}{2}\int gLg\ dx - \frac{\theta^3}{3}\int g^3\ dx\right]_{\theta=1} \\ &= \int gLg\ dx - \int g^3\ dx = 2E(g) - \frac{1}{3}\int g^3\ dx. \end{aligned}$$

But from Lemmas 3.4 and 3.5.2 we have $E(g) = I_q < 0$ and $\int g^3\ dx > 0$, so that

$$\frac{d}{d\theta}\left[E(\theta g)\right]_{\theta=1} < 0.$$

Now, using the definition of the Fréchet derivative, we have

$$\frac{d}{d\theta}\left[E(\theta g)\right]_{\theta=1} = \int \delta E(g)\cdot\frac{d}{d\theta}\left[\theta g\right]_{\theta=1}\ dx = -\lambda\int \delta Q(g)\cdot g\ dx = -\lambda\int g^2\ dx;$$

and since $\int g^2\ dx > 0$ it follows that $\lambda > 0$ as claimed.

Remark. There remains the question of whether a stability result such as Theorem 3.1 holds for solitary-wave solutions of (3.1) for arbitrary values of H_1 and H_2. As noted in Section 2, such a result could be established if it could be shown that the set G_q consists of the translates of a single function. For this, in turn, it would suffice to show that solitary-wave solutions of (3.1) are unique up to translations. (Note that in the absence of a uniqueness result, we do not even know whether the solitary waves discussed in [**ABS**] are the same as those in the sets G_q.) Such a uniqueness result has indeed been proved for the case $H_1 = H_2$ [**A2,AT**], but the existing proofs rely heavily on an algebraic property of equation (3.1) which does not hold in the case $H_1 \neq H_2$.

Alternatively, a result like Theorem 3.1 may follow from a local analysis such as that appearing in [**A1**] and [**AB**]; a major obstacle to this approach, however, is the lack of an explicit formula for solitary-wave solutions when $H_1 \neq H_2$.

4. Further results

Clearly the proof of Theorem 3.2 did not rely heavily on special properties of the operator L appearing in equation (3.1), and similar results may be obtained in more general settings. For an equation of type (1.5), for example, all the arguments used to prove Theorem 3.2 will go through without change under the following assumptions on the function $f(u)$ and the symbol $m(k)$ of the operator L:

(4.1) $$A_1|k|^s \leq m(k) \leq A_2|k|^s \text{ for all } k \geq 1,$$
where A_1, A_2 are positive constants and $s \geq 1$;

(4.2) $$m(k) \geq 0 \quad \text{for all } k \in \mathbf{R};$$

$$\sup_{k\in\mathbf{R}} |k|^n \left| \left(\frac{d}{dk}\right)^n \left(\frac{m(k)}{k}\right) \right| < \infty \quad \text{for all } n \in \mathbf{N}; \tag{4.3}$$

and either

$$f(u) = |u|^{p+1} \quad \text{for some } p \in (0, 2s) \tag{4.4a}$$

or

$$f(u) = u^{p+1} \quad \text{for some } p \in \mathbf{N} \text{ satisfying } p < 2s. \tag{4.4b}$$

Assumptions (4.1) and (4.4) are used in proving analogues of Lemmas 3.4 and 3.5.1; the condition $p < 2s$ guarantees that the Sobolev embedding theorem can be used in these proofs in the same way as in Section 3. The proof of Lemma 3.5.2 made use of the positivity of the symbol $m(k)$, and assumption (4.2) guarantees that the same proof applies here. Finally, assumption (4.3), or some such condition on the regularity and decay of $m(k)$, is needed to prove an analogue of Lemma 3.7. As the proof of Lemma 3.7 shows, (4.3) implies that the operator L is nearly a local operator, in the sense that Lu is small at points far away from the support of u.

From these observations there results the following theorem on stability of solitary-wave solutions of (1.5):

THEOREM 4.1. *Suppose that assumptions (4.1), (4.2), (4.3) and either (4.4a) or (4.4b) above hold for the functions $f(u)$ and $m(k)$. Then for each $q > 0$ there exists a nonempty set G_q, consisting of solitary-wave solutions g of (1.5) having positive wavespeeds and satisfying $Q(g) = q$, which is stable in the following sense: for every $\epsilon > 0$ there exists $\delta > 0$ such that if*

$$\inf_{g\in G_q} \|u_0 - g\|_{s/2} < \delta,$$

then the solution $u(x,t)$ of (1.5) with $u(x,0) = u_0$ satisfies

$$\inf_{g\in G_q} \|u(\cdot,t) - g\|_{s/2} < \epsilon$$

for all $t \in \mathbf{R}$.

Examples of equations to which the assumptions of Theorem 4.1 apply are the Benjamin-Ono equation [**B1**] and the Smith equation [**S**], which correspond to the dispersion operators with symbols $m(k) = |k|$ and $m(k) = \sqrt{1+k^2}-1$, respectively. For the Benjamin-Ono equation in the case $f(u) = u^2$, there is a unique solitary-wave solution (up to translation) [**AmT1**], and hence in this case the stability result of Theorem 4.1 implies a stronger result like that of Theorem 1.1. Such a result is already known (cf. [**BBSSS**]), but for the Benjamin-Ono equation with other nonlinearities $f(u)$ and for the Smith equation, Theorem 4.1 appears to be the only stability result available to date.

For the fifth-order KdV-type equation

$$u_t + u^p u_x + u_{xxx} - \delta u_{xxxxx} = 0, \tag{4.5}$$

assumptions (4.1) through (4.4) hold with $s = 4$ if $p < 8$ and $\delta > 0$, and so in this case Theorem 4.1 implies existence of non-empty stable sets of solitary-wave solutions G_q for each $q > 0$. Each element g of G_q, being a solitary-wave solution of (4.5) with positive wavespeed, will satisfy the equation

$$Cg - g'' + \delta g'''' = \frac{g^{p+1}}{p+1}, \tag{4.6}$$

for some $C > 0$. In their study [**AmT2**] of equation (4.6) in the case $p = 1$, Amick and Toland showed that for all $C > 0$ and $\delta > 0$, even solutions $g(x)$ exist which satisfy $g(0) > 0$ and which decay to zero as $|x| \to \infty$; the decay being monotonic in $|x|$ if $C \leq 1/(4\delta)$ and oscillatory otherwise. We do not know which (if any) of these solutions of (4.6) are also contained in the stable sets G_q.

An interesting question is whether a result such as Theorem 4.1 holds for equations in which the symbol $m(k)$ is not everywhere positive, i.e., when assumption (4.2) does not hold. An equation of possible physical interest in which this situation obtains is the Benjamin equation

$$u_t + uu_x + \mathcal{H}u_{xx} + \delta u_{xxx} = 0,$$

derived in [**B3**] as a model for interfacial waves in a stratified fluid in the presence of strong surface tension. Here $\mathcal{H}$ denotes the Hilbert transform, $\delta > 0$, and the equation fits the form (1.5) with symbol $m(k) = -|k| + \delta k^2$. It is shown in [**ABR**] that for large values of C, there exist oscillatory solitary-wave solutions $u(x,t) = \phi(x - Ct)$ which have a strong stability property like that expressed in Theorem 1.1; but for smaller values of C (which are the values of physical interest), it remains an open question whether stable solitary-wave solutions exist. Similarly, for the equation

$$u_t + uu_x - u_{xxx} - \delta u_{xxxxx} = 0,$$

with $\delta > 0$, Amick and Toland proved in [**AmT2**] the existence of oscillatory solitary waves at least for some values of the wavespeed C, but again Theorem 4.1 does not apply because the associated symbol $m(k) = -k^2 + \delta k^4$ is not everywhere positive.

Finally, another interesting direction for generalization of the above arguments would be to systems of nonlocal equations, such as those derived by Liu, Kubota, and Ko for modelling interactions between waves on neighboring pycnoclines within a stratified fluid (cf. [**ABS,LKK**]).

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