Theorem. Suppose A is an $m \times n$ matrix. Then every vector in the null space of A is orthogonal to every vector in the column space of A^T, with respect to the standard inner product on R^n.

Proof. Suppose $\mathbf{u}$ is in the null space of A and $\mathbf{v}$ is in the column space of A^T.

Since A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, which means that A^T has m columns. Let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ stand for the column vectors of A^T, so that

$$A^T = [\, \mathbf{w}_1 \quad \mathbf{w}_2 \quad \dots \quad \mathbf{w}_m \,].$$

Since $\mathbf{v}$ is in the column space of A^T, then $\mathbf{v}$ is a linear combination of the column vectors of A^T, which means that

$$\mathbf{v} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \dots a_m\mathbf{w}_m,$$

where $a_1, a_2, \dots, a_m$ are real numbers. But by definition of matrix multiplication this means that

$$\mathbf{v} = [\, \mathbf{w}_1 \quad \mathbf{w}_2 \quad \dots \quad \mathbf{w}_m \,] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}.$$

If we let $\mathbf{b}$ stand for $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$ then we can rewrite the last equation as

$$\mathbf{v} = A^T\mathbf{b}.$$

Now to prove the theorem we write:

$$\begin{aligned}
(\mathbf{u}, \mathbf{v}) &= \mathbf{u}^T\mathbf{v} && \text{(by definition of standard inner product on } \mathbf{R}^n\text{)} \\
&= \mathbf{u}^T A^T \mathbf{b} && \text{(since } v = A^T\mathbf{b}\text{)} \\
&= (A\mathbf{u})^T\mathbf{b} && \text{(since } \mathbf{u}^T A^T = (A\mathbf{u})^T\text{, by properties of transpose)} \\
&= \mathbf{0}^T\mathbf{b} && \text{(since } A\mathbf{u} = \mathbf{0}\text{, because } \mathbf{u} \text{ is in the null space of } A\text{)} \\
&= 0.
\end{aligned}$$

This shows that $(\mathbf{u}, \mathbf{v}) = 0$, or in other words that $\mathbf{u}$ is orthogonal to $\mathbf{v}$, which is what we wished to prove.

Remark: In class, I stated that "every vector in the null space of A is orthogonal to every vector in the row space of A". The problem with that statement is that vectors in the null space of A are column vectors in R^n, and vectors in the row space of A are row vectors in R_n. Up to now, we have only defined the meaning of the phrase "$\mathbf{u}$ is orthogonal to $\mathbf{v}$"when $\mathbf{u}$ and $\mathbf{v}$ are a pair of vectors in the same vector space. What would it mean for a vector in one vector space, R^n, to be orthogonal to a vector in a different vector space, R_n?

You could get around this problem by defining a vector $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ in R^n to be orthogonal to a vector $\mathbf{v} = [\, v_1 \quad \dots \quad v_n \,]$ in R_n if the matrix product $\mathbf{vu} = v_1u_1 + v_2u_2 + \cdots + v_nu_n$ is equal to 0. Then to prove the theorem as I stated it in class, you would have to show that for every vector $\mathbf{u}$ in the null space of A and every $\mathbf{v}$ in the row space of A, we have $\mathbf{vu} = 0$. In fact I did prove the theorem this way in the 1:30 section.