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Math 4513

Senior Math Seminar

with Prof. John Albert

A polynomial spherical function is a polynomial which satisfies the Laplace equation.

The Laplace equation is: (in 3 dimensions)

$$\nabla^2 f(x, y, z) = 0 \quad \text{or} \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

which is often notated $\Delta f = 0$. The unsolved question we are looking at deals with the number of sections ~~on~~ zeros of the n^{th} degree polynomial subdivides a sphere. If we look at homogeneous polynomials, (homogeneous meaning all terms are of the same degree) for reasons that will be elaborated on later, then our first degree polynomial that satisfies the laplace equation is:

$$f_1(x, y, z) = \alpha_1 x + \alpha_2 y + \alpha_3 z$$

where α_1, α_2, and α_3 are arbitrary constants.

Note that this is a solution because $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} = 0$.

To find a second-degree polynomial that satisfies the laplace equation we must find an $f_2(x,y,z)$

$$f_2(x,y,z) = \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2 + \alpha_4 xy + \alpha_5 xz + \alpha_6 yz$$

that satisfies $\nabla^2 f_2 = 0$.

Note that α_4, α_5, and α_6 may be any real constant and the laplace equation still holds so:

$$\nabla^2 f_2 = \frac{\partial^2}{\partial x^2} f_2 + \frac{\partial^2}{\partial y^2} f_2 + \frac{\partial^2}{\partial z^2} = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 0$$

So our general function (2nd degree polynomial) which satisfies Laplace's equation is

$$f_2(x,y,z) = \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2 + \alpha_4 xy + \alpha_5 xz + \alpha_6 yz$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and $\alpha_4, \alpha_5, \alpha_6 \in \mathbb{R}$.

To find a polynomial of degree three which satisfies Laplaces equation we go through the same procedures.

$$\left.\begin{aligned} f_3(x,y,z) = \alpha_1 x^3 &+ \alpha_2 y^3 + \alpha_3 z^3 \\ &+ \alpha_4 x^2 y + \alpha_5 x^2 z \\ &+ \alpha_6 y^2 x + \alpha_7 y^2 z \\ &+ \alpha_8 z^2 x + \alpha_9 z^2 y \\ &+ \alpha_{10} x y z . \end{aligned}\right\} \quad (1)$$

$$\frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2} =$$

$$\begin{aligned} = \; & 6\alpha_1 x + 2\alpha_4 y + 2\alpha_5 z \\ & + 6\alpha_2 y + 2\alpha_6 x + 2\alpha_7 z \\ & + 6\alpha_3 z + 2\alpha_8 x + 2\alpha_9 y = 0 \end{aligned}$$

$\Rightarrow$ either $x, y, z = 0$ or:

$$\left|\begin{aligned} 6\alpha_1 + 2\alpha_6 + 2\alpha_8 &= 0 \\ 2\alpha_4 + 6\alpha_2 + 2\alpha_9 &= 0 \\ 2\alpha_5 + 2\alpha_7 + 6\alpha_3 &= 0 \end{aligned}\right.$$

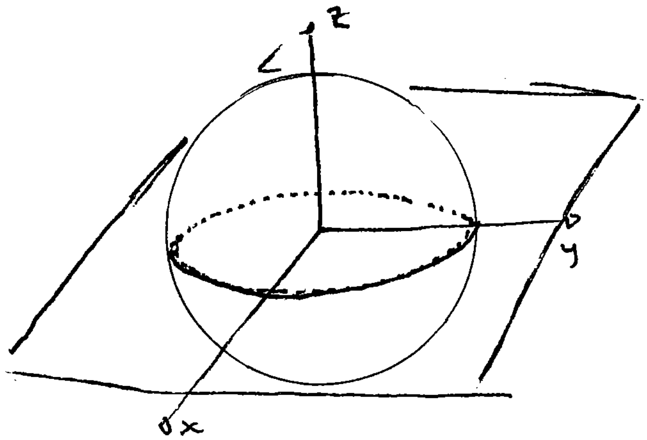
$$\Rightarrow \left|\begin{aligned} 3\alpha_1 + \alpha_6 + \alpha_8 &= 0 \\ 3\alpha_2 + \alpha_4 + \alpha_9 &= 0 \\ 3\alpha_3 + \alpha_5 + \alpha_7 &= 0 \end{aligned}\right. \quad (1a)$$

So any polynomial of the form (1) satisfying conditions (1a) where $\alpha_{10} \in \mathbb{R}$ will satisfy Laplace's equation.

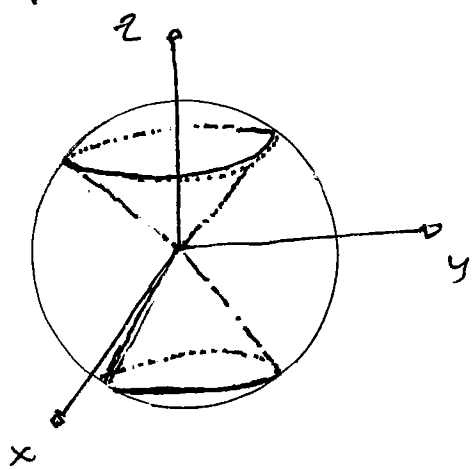
Note: also we would say a zero degree polynomial satisfies the laplace equation but it seems pretty obvious.

One might ask the question as mentioned previously: into how many sections do these polynomials sub-divide a sphere?

Our first degree equation is easy... it's a plane that goes through the origin so it divides a sphere into two parts:



Our second degree polynomial, if we assume $\alpha_4 = \alpha_5 = \alpha_6 = 0$ if $f_2(x, y, z) = \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2$ where $\alpha_1 + \alpha_2 + \alpha_3 = 0$ looks like this and divides (the surface of) a sphere into 3 parts.



As you might recall the expression for our 3^{rd} degree polynomial was a little horrific and I don't want to figure out how many places it divides a sphere. It motivates the question at hand which is how many sections does an n^{th} degree polynomial (homogeneous) spherical function zeros of the subdivide the surface of a sphere. Apparently no one really knows but this guy named Courant proved that the n^{th} eigenfunction has at most n nodal domains. (Nodal domains are domains where the function is zero along the boundary). What does this have to do with our question? First we need to understand why $\nabla^2 f(x,y,z) = 0$ is an eigen function.

To do this we make a simple change of variables letting $f(x,y,z) \longrightarrow u(r,\theta,\phi)$

$$
\begin{aligned}
x &\longmapsto r\sin\phi\cos\theta \\
y &\longmapsto r\sin\phi\sin\theta \qquad (2) \\
z &\longmapsto r\cos\phi
\end{aligned}
$$

This is just a standard change to spherical coordinates where I'm using the convention that ϕ is the azimuthal (from $+z$) angle, and θ is the angle in the xy plane from the positive x axis. These expressions for $x, y, \& z$ can be obtained with simple trig and I won't go through it here.

Figuring out what the ∇^2 operator is in spherical coordinates is not quite as easy though. It seems that there are three ways to go about this: one involves something called "line elements" but I don't understand it and it doesn't have a proof associated with it in the books I've read, the second involves the general form of the Laplacian in arbitrary coordinates but I found it on the internet and I didn't see a proof so I'll outline my own way using the good old chain rule which should be correct.

what we are looking for is a way to write

$\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 u}{\partial z^2}$ in terms of r, θ, and ϕ.

To do this you first set up the three equations: (by chain rule)

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} \qquad (3)$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \phi}$$

note that the only thing not in terms of r, θ and ϕ here are u_x, u_y, and u_z, because

(from (2))

$$\frac{\partial x}{\partial r} = \sin\phi\cos\theta \qquad \frac{\partial y}{\partial r} = \sin\phi\sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\phi\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\sin\phi\cos\theta$$

$$\frac{\partial x}{\partial \phi} = r\cos\phi\cos\theta \qquad \frac{\partial y}{\partial \phi} = r\cos\phi\sin\theta$$

$$\frac{\partial z}{\partial r} = \cos\phi$$

$$\frac{\partial z}{\partial \theta} = 0$$

$$\frac{\partial z}{\partial \phi} = -r\sin\phi$$

So (3) becomes:

$$u_r = u_x\sin\phi\cos\theta + u_y\sin\phi\sin\theta + u_z\cos\phi$$

$$u_\theta = u_x(-r\sin\phi\sin\theta) + u_y r\sin\phi\cos\theta + u_z(0)$$

$$u_\phi = u_x r\cos\phi\cos\theta + u_y r\cos\phi\sin\theta + u_z(-r\sin\phi)$$

This is a set of 3 equations with three unknowns

so it can be written as

$$\begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_r \\ u_\theta \\ u_\phi \end{bmatrix}$$

So the system

$$\left[\begin{array}{ccc:c} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi & u_r \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 & u_\theta \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi & u_\phi \end{array}\right]$$

can be solved by elementary row operations to obtain $u_x(r,\theta,\phi)$, $u_y(r,\theta,\phi)$ and $u_z(r,\theta,\phi)$. Now, once we've done that you take those expressions and use the chain rule again so that you get 9 equations:

$$\frac{\partial u_x}{\partial r} = \frac{\partial u_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u_x}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u_x}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial u_x}{\partial \theta} = \frac{\partial u_x}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u_x}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u_x}{\partial z}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial u_x}{\partial \phi} = \frac{\partial u_x}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u_x}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial u_x}{\partial z}\frac{\partial z}{\partial \phi}$$

$$\frac{\partial u_y}{\partial r} = \frac{\partial u_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u_y}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u_y}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial u_y}{\partial \theta} = \frac{\partial u_y}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u_y}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u_y}{\partial z}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial u_y}{\partial \phi} = \frac{\partial u_y}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u_y}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial u_y}{\partial z}\frac{\partial z}{\partial \phi}$$

$$\frac{\partial u_z}{\partial r} = \frac{\partial u_z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u_z}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u_z}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial u_z}{\partial \theta} = \frac{\partial u_z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u_z}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u_z}{\partial z}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial u_z}{\partial \phi} = \frac{\partial u_z}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u_z}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial u_z}{\partial z}\frac{\partial z}{\partial \phi}$$

These equations, along with (3), if I've used the chain rule properly are sufficient to give you

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$ in terms of r, θ, and ϕ.

If you do this then you should come up with the equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\, u_\phi) + \frac{1}{r^2 \sin^2\phi} u_{\theta\theta} = 0$$

we can get rid of the r^2:

$$\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\, u_\phi) + \frac{1}{\sin^2\phi} u_{\theta\theta} = 0 \qquad (4)$$

Now we will make the assumption that our function $u(r, \theta, \phi)$ can be written as:

$$u(r, \theta, \phi) = v(r)\, Y(\theta, \phi)$$

A product of two functions, one depending only on r and one depending only on θ and ϕ. To introduce some notation, I will write:

$$u_r = \frac{\partial}{\partial r}[v(r)\, Y(\theta, \phi)] = v' Y$$

$$u_\theta = \frac{\partial}{\partial \theta}[v(r)\, Y(\theta, \phi)] = v Y_\theta$$

$$u_\phi = \frac{\partial}{\partial \phi}[v(r)\, Y(\theta, \phi)] = v Y_\phi$$

and so on, so that our equation (4) becomes:

$$\frac{\partial}{\partial r}(r^2 v' Y) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, v Y_\phi) + \frac{1}{\sin^2\phi} v Y_{\theta\theta} = 0$$

because v is dependent only on r and Y only on θ and ϕ we can write:

$$Y\frac{\partial}{\partial r}(r^2 v') + \frac{v}{\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, Y_\phi) + \frac{v}{\sin^2\phi} Y_{\theta\theta} = 0$$

now if we multiply everything by $\frac{1}{vY}$ we get

$$\underbrace{\frac{1}{v}\frac{\partial}{\partial r}(r^2 v')}_{\text{depends only on } r} + \underbrace{\frac{1}{Y\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, Y_\phi) + \frac{1}{Y\sin^2\phi} Y_{\theta\theta}}_{\text{depends only on } \theta \text{ and } \phi} = 0$$

we can write:

$$\frac{1}{v}\frac{\partial}{\partial r}(r^2 v') = -\left[\frac{1}{Y\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, Y_\phi) + \frac{1}{Y\sin^2\phi} Y_{\theta\theta}\right]$$

This is true only if

$$\frac{1}{v}\frac{\partial}{\partial r}(r^2 v') = \lambda \qquad (5)$$

$$-\left[\frac{1}{Y\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, Y_\phi) + \frac{1}{Y\sin^2\phi} Y_{\theta\theta}\right] = \lambda \qquad (6)$$

for some values of λ

From (5) we get the eigen value equation:

$$\frac{d}{dr}(r^2 v') - \lambda v = 0 \qquad (7)$$

and from (6) the eigen value equation:

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, Y_\phi) + \frac{1}{\sin^2\phi} Y_{\phi\phi} + Y\lambda = 0 \,. \qquad (8)$$

sloppy $Y_{\theta\theta}$ again!

From this it's clear why Courant's theorem is relevant, because $\nabla^2 f = 0$ can be written as an eigen value equation as we've just shown. More over, you can take a polynomial that satisfies the Laplace equation, write it in spherical coordinates using the transformation (2) and it gives you a solution to the eigen value problems (7) & (8). The degree of the polynomial corresponds to a particular eigen value as I'll show next.

Because our polynomials are ~~\$~~ homogeneous (all terms are of same degree) and because x, y, and z all have a term of r in them. the j^{th} degree homogeneous polynomial can be written as

$$r^j Y(\theta, \phi)$$

So that $v(r) = r^j$ for any homogeneous polynomial solution that is transformed into polar coordinates.

Using (7) we obtain:

$$\frac{d}{dr}(r^2 v'(r)) - \lambda v(r) = 0$$

$$\frac{d}{dr}(r^2 (j) r^{j-1}) - \lambda r^j = 0$$

$$j \frac{d}{dr}(r^{j+1}) - \lambda r^j = 0$$

$$j(j+1) r^j - \lambda r^j = 0$$

$$\lambda = j(j+1)$$

So the j^{th} degree polynomial corresponds to the eigenvalue $\lambda = j(j+1)$. According to Courant pp. (317) there are $2j+1$ eigenfunctions that correspond to the eigenvalue $\lambda = j(j+1)$.

To collect our information:

j	λ_j	# of eigen-functions associated	eigen-function number	Courant's upper bound	actually divides a sphere into
0	0	1	1	1	1
1	2	3	2-4	2	2
2	6	5	5-9	5	3
3	12	7	10-16	10	?
⋮	⋮	⋮	⋮	⋮	?

j = degree of polynomial

$\lambda_j = j(j+1)$ j^2+j (the eigen value associated with a polynomial of degree j)

# of eigen functions associated with the eigen value $\lambda_j = 2j+1$ (By Courant)

Or in perhaps an easier to read form on the following page:

	eigenvalue λ_n	eigenfunction associated (u_n)	Courant's up. bound	
1	λ_0	u_0	1	degree zero polynomial
3	λ_1	u_1	2	
	λ_1	u_2	3	
	λ_1	u_3	4	degree 1 polynomials
5	λ_2	u_4	5	
	λ_2	u_5	6	
	λ_2	u_6	7	
	λ_2	u_7	8	
	λ_2	u_8	9	degree 2 polynomials
7	λ_3	u_9	10	
	λ_3	u_{10}	11	
	λ_3	u_{11}	12	
	λ_3	u_{12}	13	
	λ_3	u_{13}	14	
	λ_3	u_{14}	15	
	λ_3	u_{15}	16	degree 3 polynomials

So we see that for a polynomial of degree j Courant's upper bound C_j gives us:

$$C_j = \sum_{k=0}^{j-1}(2k+1) + 1 \quad , \quad C_0 = 1$$

$$\boxed{C_j = j^2 + 1}$$

Courant's upper bound for a polynomial of degree j

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