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Presentation: Spherical Coordinates/Laplace Equation

The question is "What is the largest number of parts into which the zeros of an n-th degree polynomial spherical function, that satisfies the Laplace Equation, subdivides a sphere? What is the largest number of maxima of such a function?"

First of all, a Laplace Equation is a partial differential equation that was named after its creator, Pierre-Simon Laplace, a French mathematician and astronomer. A solution of a Laplace Equation is called a harmonic function. These solutions are important to many fields of science including electromagnetism, astronomy, and fluid dynamics because they describe the behavior of electric, gravitational and fluid potentials. A Laplace Equation can be written in many forms, most notably $\nabla^2 f = 0$. For this problem it will be helpful to recognize it in another form, $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$. The ∇^2 is called the Laplacian. "The Laplace operator is a second order differential operator in the n-dimensional Euclidean space, defined as the divergence of the gradient." ("Laplace Operator", Wikipedia)

For this question we need to look for functions that satisfy the above Laplace Equation with the parameters that it must be a homogeneous polynomial. It must be a homogeneous polynomial because, if it is not, the variables are harder to separate. If homogeneous, it's easier to split into an eigenvalue problem. A homogeneous polynomial is one whose terms all have the same degree. For example $x^5 + 2x^3y^2 + 9xy^4$ is a homogeneous polynomial of degree 5 because as you can see, the sum of all the exponents is 5.

I will now give examples of a 0, 1, and 2 degree homogeneous polynomial and explain how they satisfy the Laplace Equation.

<u>0 Degree Homogeneous Polynomial</u>

These are usually in the form f(x,y,z)=k where k is a constant. This equation needs to satisfy the Laplace Equation by equaling 0.

$$\nabla^2 f = 0$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Taking the first derivative of a constant gives 0; taking the second derivative also gives 0 so this does satisfy the Laplace Equation.

1 Degree Homogeneous Polynomial

These are usually in the form f(x, y, z)= kx+fy+gz where k, f and g are constants. This equation needs to satisfy the Laplace Equation by equaling 0.

The first step is to take the entire equation and differentiate with respect to x. This leaves a k because the other terms have gone to zero; then differentiate again and you are left with a 0. Take the entire equation again and differentiate with respect to y, which leaves only f. Then differentiate again and you get a 0. Take the entire equation again and differentiate with respect to z, which leaves a z. Differentiate a second time and this leaves 0 proving that this equation satisfies the Laplace Equation.

2 Degree Homogeneous Polynomial

These are usually in the form f(x,y,z)= $kx^2+fy^2+gz^2$ This equation needs to satisfy the Laplace Equation by equaling 0. An example of a 2nd degree homogeneous polynomial might be $f(x,y,z)=\alpha_1x^2+\alpha_2y^2+\alpha_3z^2+\alpha_4xy+\alpha_5xz+\alpha_6yz$. These are the only possible 2nd degree terms you can come up with and for this we are trying to find constants to satisfy the Laplace Equation. So plug this equation above into the Laplace Equation:

$$\frac{\partial}{\partial x^2}[\alpha_1x^2+\alpha_2y^2+\alpha_3z^2+\alpha_4xy+\alpha_5xz+\alpha_6yz+]+\frac{\partial}{\partial y^2}[\alpha_1x^2+\alpha_2y^2+\alpha_3z^2+\alpha_4xy+\alpha_5xz+\alpha_6yz]+$$

$$\frac{\partial}{\partial z^2}[\alpha_1x^2+\alpha_2y^2+\alpha_3z^2+\alpha_4xy+\alpha_5xz+\alpha_6yz]=0$$

After taking the derivatives of all of this, only one equation remains

$2\alpha_1+2\alpha_2+2\alpha_3=$ 0=0.

Dividing through by 2 gives $\alpha_1+\alpha_2+\alpha_3=0$. Alpha 4,5 and 6 all go to zero so they can be any number. So we're looking for polynomials of the above form

$f(x,y,z)=\alpha_1x^2+\alpha_2y^2+\alpha_3z^2+\alpha_4xy+\alpha_5xz+\alpha_6yz$ where $\alpha_1+\alpha_2+\alpha_3=0$

Spherical Coordinates

The standard spherical coordinates are

$x=\rho\sin\phi\cos\vartheta$

$y=\rho\sin\phi\sin\vartheta$

$z=\rho\cos\phi$

where

ρ is the radial distance from a point to the origin

ϕ is called the zenith or a polar angle from the z axis

θ is called the azimuthal angle and is located from the positive xz plane to the point

Bibliography

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