Midterm review

The midterm will consist of five or six short problems related to the topics covered in lectures during the first half of the course. For your convenience I've listed most of these topics below. The list includes topics covered by the first three assignments, but does not include Pythagorean triples, rational solutions of algebraic equations (Diophantine problems), or the problem of finding triangles with rational sides and rational area. These will not be covered on the midterm, but will be covered on a future assignment (assignment 4).

The list is mainly intended to help you by summarizing what we've done in the course so far. You won't need to repeat any of the proofs on the exam. It should be enough to have sat in the lectures and been able to follow the proofs as they were presented, and to have done the homework, or at least given it a good try. If as you're going down the list you see a topic that doesn't look familiar, then you should probably try to brush up on it.

- Proof of the irrationality of $\sqrt{2}$ using the principle of infinite descent.
- Euclid's proof that there are infinitely many primes.
- Euclidean algorithm for finding the greatest common divisor of two numbers.
- The relationship between the Euclidean algorithm and "anthyphairesis", a fancy word for "chopping squares off a rectangle".
- Proof of the irrationality of $\sqrt{2}$ using anthyphairesis.
- Proof, using the Euclidean algorithm, that the greatest common divisor of two numbers a and b can be written in the form $ax + by$, for some integers x and y. For want of a better name, let's call this the "g.c.d. theorem".
- Proof, using the g.c.d. theorem, of the "prime divisor property": if a prime p divides ab then p must divide either a or b.
- Proof, using the prime divisor property, of the "unique prime factorization" theorem for natural numbers (also called the "fundamental theorem of arithmetic"). This theorem says that any natural number can be written as a product of primes in a unique way. More precisely, there is essentially only one way to write a natural number n as a product of primes, $n = p_1 p_2 p_3 \cdots p_k$ (where the primes $p_1, p_2, \ldots, p_k$ are not necessarily distinct). The only other ways to write n as a product of primes would be to multiply together the same primes $p_1, p_2, \ldots, p_k$ in a different order, and of course this is not really a different factorization of n.
- Proof of the irrationality of $\sqrt{2}$ using the fundamental theorem of arithmetic.
- Euclid's proof that if p is a prime of the form $p = 2^n - 1$, then the number $2^{n-1}p$ is perfect.
- Proof of the converse of the preceding theorem: if N is an even perfect number, then N is of the form $2^{n-1}p$, where $p = 2^n - 1$ and p is prime.
- Ruler and compass constructions. Constructible points in the plane.
- Proof that any constructible point has coordinates obtainable by starting with the natural numbers and performing finitely many of the following operations: addition, subtraction, multiplication, division, and square root.
- Proof of the converse of the preceding result: any point with coordinates obtainable from the natural numbers by the above operations is a constructible point.
- Proof that $\sqrt[3]{2}$ is not constructible.
- Newton's construction of $\sqrt[3]{2}$ using a ruler and a marked straightedge. (This is called a "neusis" construction.)
- Euclid's proof of the Pythagorean theorem (which was done purely geometrically, without using any multiplication or addition of lengths of segments).
- Proof that the central angle subtended by an arc of a circle is twice the angle subtended by the arc at the circumference of the circle.
- The Greeks defined two polygons to have equal area if one could be cut into pieces which could be reassembled to form the other. Two such polygons are said to be "equidecomposable". It seems obvious that if one polygon P_1 contains a smaller polygon P_2, then P_1 should not be equidecomposable with P_2; i.e., a whole and its part should not be equidecomposable. However, the Greeks were never able to

prove this (and not for want of trying). Today, we settle the issue by defining area in terms of numbers: first we define the area of a triangle to be one half its base times its height, and then we define the area of a general polygon to be the number obtained by cutting the polygon into triangles and adding up the areas of the triangles. If you want to get away with this definition, however, you have to prove that it leads to a consistent result. First you have to check that when computing the area of a triangle as one-half the base times the height, it doesn't matter which side you use as base. We did this in class. Next, you need to prove that no matter how you cut a polygon up into triangles, when you add up the areas of the triangles you always get the same number. We also did this in class, following a proof given by Hilbert in 1900.

- Although the modern definition of area is more convenient than the Greek definition, it's not immediately obvious that the two definitions are equivalent. In other words, it has to be proved that if two polygons have the same area according to the modern definition (i.e., when you cut them into triangles and add up the areas of the triangles, you get the same number for both polygons), then they have the same area according to the Greek definition (i.e., they are equidecomposable). We did part of this proof in class and part in the homework (Assignment 3).
- The method the Greeks used to define area doesn't work so well for defining volumes. The problem is that two polyhedra (solid objects with flat faces) with equal volumes might not be equidecomposable. In fact, Hilbert's student Dehn proved that a cube and a regular tetrahedron of equal volume are not equidecomposable; you cannot cut a cube up into finitely many pieces and reassemble them to form a regular tetrahedron (or vice versa). So, to prove the formula for the volume of a regular tetrahedron (volume equals one-third the area of the base times the height), we need to resort to *infinite* processes, such as infinite series, or derivatives or integrals (which are infinite processes in the sense that they are defined in terms of limits). In class we showed how to compute the volume of a regular tetrahedron by chopping it into infinitely many prisms and computing the sum of the volumes of the prisms as an infinite geometric series.