Review for First Exam (part 1)

The first exam will cover sections 7.2, 7.3, 7.4, 7.6, 7.7, and 7.8 of the text. Here is a summary of things you should know for the exam from sections 7.2, 7.3, and 7.4.

1. If $a > 0$ and m is a natural number ($m = 1, 2, 3, \dots$), then a^m means a multiplied by itself m times. If m is not a natural number, for example $m = \frac{1}{2}$, or $m = 0$, or $m = -1$, or $m = \sqrt{5}$, then a^m is defined in such a way that the following laws of exponents hold:

$$\begin{aligned} a^{m+n} &= a^m \cdot a^n \\ a^{m-n} &= \frac{a^m}{a^n} \\ (a^m)^n &= a^{mn}. \end{aligned}$$

From these laws it follows that a^0 must equal 1, that a^{-1} must equal $1/a$, that $a^{1/2}$ must equal one of the square roots of a, etc. (In this class we always define $a^{1/2}$ to be the *positive* square root of a.)

There are a couple of other important laws of exponents, but the above three are the most important.

2. If $a > 0$, then "$y = \log_a x$" , read "y is the logarithm to the base a of x", means that y is the exponent you have to raise a to, in order to obtain x. That is, $y = \log_a x$ means that $a^y = x$. Thus a logarithm is an exponent. Since the value of a^y is always positive, no matter what y is (when a is positive), it follows that $\log_a(x)$ can only be defined when x is positive. From the above three laws of exponents follow the following three laws of logarithms:

$$\begin{aligned} \log_a(mn) &= \log_a m + \log_a n \\ \log_a\left(\frac{m}{n}\right) &= \log_a m - \log_a n \\ \log_a(p^n) &= n \log_a p \end{aligned}$$

There are other laws of logarithms, but these are the most important.

3. The number e can be defined in several ways. Our textbook defines it as "the number which makes the equation

$$\lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) = 1$$

true". A more direct way to define it would be as

$$e = \lim_{h \to 0} (1 + h)^{1/h}.$$

Yet another definition of e, which has the advantage of making it easy to compute a decimal expansion of e to many decimal places of accuracy, is

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots,$$

where the "..." means that as more and more numbers are added together on the right-hand side, the resulting sum comes closer and closer to the limiting value which is e.

Of course, all three of the above definitions ultimately yield the same value of e (although it may not be so obvious why).

4. The logarithm to the base e of x is called the "natural logarithm" of x, and is written as $\ln x$ (thus $\ln x$ and $\log_e x$ are the same). In other words, $y = \ln x$ means that y is the exponent you have to raise e to, in order to obtain x. Since $\ln x$ is just a particular case of $\log_a x$, the laws of logarithms mentioned above also hold for $\ln x$.

5. If $y = e^x$, then $\frac{dy}{dx} = e^x$. This is because

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\ &= \lim_{h\to 0} e^x \frac{(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{(e^h - 1)}{h} = e^x \cdot 1 = e^x.\end{aligned}$$

You should be able to follow each step in this string of equalities.

From $\frac{d}{dx}(e^x) = e^x$ it follows that $\int e^x \, dx = e^x + C$.

6. If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$. You should be able to give a proof of this fact if asked. You can find a proof in the textbook (it starts at the bottom of p. 411 and ends at the top of p. 412).

It follows from this fact that $\int \frac{1}{x} \, dx = \ln x + C$ (provided we restrict x to have positive values). A formula which is also valid when x has negative values is

$$\int \frac{1}{x} \, dx = \ln |x| + C.$$

However, you have to be careful: this formula cannot be used to integrate $1/x$ over intervals of x which contain $x = 0$.

7. Functions which involve exponentials, including $y = a^x$, or $y = x^a$, or $y = x^x$, or $y = f(x)^{g(x)}$ for general functions $f(x)$ and $g(x)$, can be easily differentiated using "logarithmic differentiation". This process involves first taking the logarithm of the equation $y = f(x)^{g(x)}$ and then using implicit differentiation. See pp. 417–418 of the text.

8. You should be comfortable with using the chain rule and integration by substitution (these are two sides of the same coin) to differentiate and integrate functions which involve exponentials and logarithms.