Combinatorial Game Theory

Math 4513

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Combinatorial Game Theory

"Combinatorial Game Theory, or CGT, is only valid for games satisfying certain conditions. The most important are that there are two players, both players have complete knowledge about the state of the game, there is no chance element (such as a dice roll or card shuffle), and both players take turns to make a move (skipping turns is not allowed) so that the first player unable to move loses the game. So Backgammon, Poker, Battleships, Monopoly, and Solitaire, among others, cannot be considered by CGT as they fail various combinations of these criteria."
http://plus.maths.org/issue27/features/dartnell/index.html

The Combinatorial Game of Nim

"Nim is a two player mathematical game of strategy in which players take turns removing objects from distinct heaps. On each turn, a player must remove at least one object, and may remove any number of objects provided they all come from the same heap."
"The name is probably derived from German *nimm!* meaning "take!", or the obsolete English verb *nim* of the same meaning. It should also be noted that turning the word *NIM* upside-down and backwards results in *WIN*"
http://en.wikipedia.org/wiki/Nim

Nim follows the rules for combinatorial games

- There are only two players
- There is a finite set of possible positions in every game
- The rules of the game specify for both players and each position which moves to other positions are legal moves
- Players alternate moving
- The game ends when a position is reached, called a terminal position, from which no other moves from the other player can be made
- The game ends in a finite number of moves no matter how it is played
- There are no chance moves

academics.smcvt.edu/.../combo2/Archive/Combo%20s03/special%20topics%2003/Progressively%20Finite%20Games.ppt

Creating a Winning Strategy

"The key to the theory of the game is the binary digital sum of the heap sizes, that is, the sum (in binary) neglecting all carries from one digit to another. This operation is also known as exclusive or (xor) or vector addition over **GF**(2). Within combinatorial game theory it is usually called the **nim-sum**, as will be done here. The nim-sum of x and y is written $x \oplus y$ to distinguish it from the ordinary sum, $x + y$."
http://en.wikipedia.org/wiki/Nim

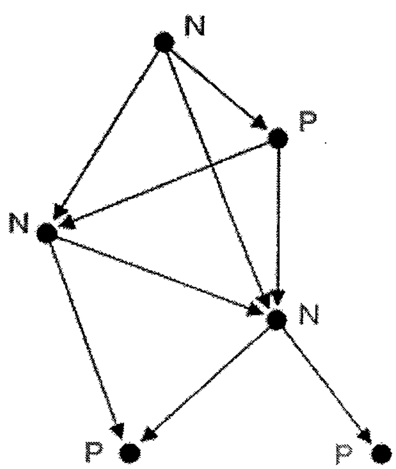
***P*- and *N*-positions**

"We can classify each position in the game according to whether it is a first- or a second-player win, if both players play optimally starting from that position. A first-player-win position is known as an *N-position* (because the ***n***ext player is to win), while a second-player-win position is known as a *P-position* (because the ***p***revious player is to win).

The *P*- and *N*-positions can be characterized inductively as follows:

- A vertex *v* is a *P*-position if and only if all its direct followers are *N*-positions.
- A vertex *v* is an *N*-position if and only if it has some *P*-position follower.

The induction starts at the sinks, which are *P*-positions because they vacuously satisfy the *P*-position requirement.



In the 1930s, the Sprague-Grundy theorem showed that all impartial games are equivalent to heaps in nim, thus showing that major unifications are possible in games considered at a combinatorial level (in which detailed strategies matter, not just pay-offs)."
http://yucs.org/~gnivasch/cgames/spraguegrundy/index.html

"So, for example, the position v = (5, 7, 2) is a P-position since 5 = 101, 7 = 111, 2 = 10, with Nim sum 0. It helps do the calculation to organize the numbers in a table like this

1 0 1	5
1 1 1	7
0 1 0	2
0 0 0	0

A position that is not a P-position is called an N-position (N for next player wins); hence, every position in the game is either P or N.
What we will show is that these positions have the property that
(1) Any move from a P-position ends in an N-position.
(2) From an N-position there is at least one move that ends in a P-position.

Granted this, the strategy to win the game is to always move to a P-position (whether you can get this started or not depends on the original position of the game if you are the first player and otherwise on how your opponent plays!). The reason is that by moving to a P-position you force your opponent to move to an N-position, from which you can then again move to a P-position and so on until the game ends. (The game ends in finitely many steps, as the total number of chips decreases at every move.) The end position, when no chips are left, is a P-position and hence it must be you who got there, winning the game.
In order to prove the two crucial statements about the N and P-positions we need to look at Nim addition a bit more closely. First, it is not hard to verify that _ is a commutative and associative operation on the numbers 0, 1, 2 · · · and hence behaves like ordinary sum except for the unusual property that for any n"
www.ma.utexas.edu/~villegas/S05/handout-5.pdf

Conversely, the position v = (4,5,6) is an N position since the nim sum of the number of objects in each pile does equal 0. In this instance the first player has the distinct advantage.

Proof of Strategy

"**Theorem**. In a normal Nim game, the first player has a winning strategy if and only if the nim-sum of the sizes of the heaps is nonzero. Otherwise, the second player has a winning strategy.

Proof: Notice that the nim-sum ($\oplus$) obeys the usual associative and commutative laws of addition (+), and also satisfies an additional property, $x \oplus x = 0$ (technically speaking, the nonnegative integers under $\oplus$ form an Abelian group of exponent 2).

Let $x_1, \ldots, x_n$ be the sizes of the heaps before a move, and $y_1, \ldots, y_n$ the corresponding sizes after a move. Let $s = x_1 \oplus \ldots \oplus x_n$ and $t = y_1 \oplus \ldots \oplus y_n$. If the move was in heap k, we have $x_i = y_i$ for all $i \neq k$, and $x_k > y_k$. By the properties of $\oplus$ mentioned above, we have

$$
\begin{aligned}
t &= 0 \oplus t \\
&= s \oplus s \oplus t \\
&= s \oplus (x_1 \oplus \ldots \oplus x_n) \oplus (y_1 \oplus \ldots \oplus y_n) \\
&= s \oplus (x_1 \oplus y_1) \oplus \ldots \oplus (x_n \oplus y_n) \\
&= s \oplus 0 \oplus \ldots \oplus 0 \oplus (x_k \oplus y_k) \oplus 0 \oplus \ldots \oplus 0 \\
&= s \oplus x_k \oplus y_k
\end{aligned}
$$

(*) $t = s \oplus x_k \oplus y_k$.

The theorem follows by induction on the length of the game from these two lemmata.

Lemma 1. If $s = 0$, then $t \neq 0$ no matter what move is made.

Proof: If there is no possible move, then the lemma is vacuously true (and the first player loses the normal play game by definition). Otherwise, any move in heap k will produce $t = x_k \oplus y_k$ from (*). This number is nonzero, since $x_k \neq y_k$.

Lemma 2. If $s \neq 0$, it is possible to make a move so that $t = 0$.

Proof: Let d be the position of the leftmost (most significant) nonzero bit in the binary representation of s, and choose k such that the dth bit of x_k is also nonzero. (Such a k must exist, since otherwise the dth bit of s would be 0.) Then letting $y_k = s \oplus x_k$, we claim that $y_k < x_k$: all bits to the left of d are the same in x_k and y_k, bit d decreases from 1 to 0 (decreasing the value by 2^k), and any change in the remaining bits will amount to at most 2^k-1. The first player can thus make a move by taking $x_k - y_k$ objects from heap k, then

$t = s \oplus x_k \oplus y_k$ (by (*))

$= s \oplus x_k \oplus (s \oplus x_k)$

$= 0$."

http://en.wikipedia.org/wiki/Nim

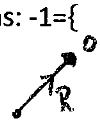
[redacted]:
Game Trees
Surreal Numbers
Domineering—an example of how numbers relate to games

Game Trees
0^{th} day games:
Endgame: 2^{nd} player wins

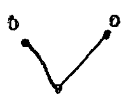
1 day games:
Endgame: 2^{nd} player wins

Left wins: 1={0| }

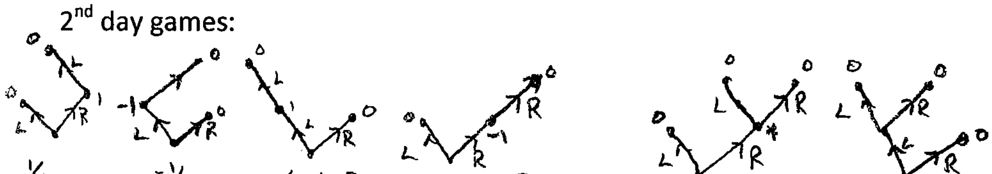
Right wins: -1={ |0}



1^{st} player wins: *={0|0}



2^{nd} day games:



$\frac{1}{2}$ $-\frac{1}{2}$ $\{1|0\}$ $\{0|-1\}$ $\uparrow=\{0|*\}$ $\downarrow=\{*|0\}$

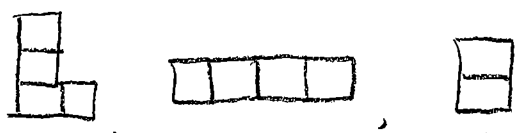
Numbers (Surreal numbers):
X=$\{L|R\}$ where x(L) ≤ x(R)
x≥y iff no x(R)≤y and x≤ no y(L)
x=y iff x≥y and y≥x
x>y iff x≥y and y is not greater than or equal to x
x+y = $\{x(L)+y, x+y(L)|x(R)+y, x+y(R)\}$
-x = $\{-x(R)|-x(L)\}$
xy=
$\{x(L)y+xy(L)-x(L)y(L), x(R)y+xy(R)-x(R)y(R)|x(Ly+xy(R)-x(Ly(L), x(Ry+xy(L)-x(Ry(L)\}$

Examples:

-1=$\{\ |0\}$, $\{\ |0,1\}$
-1/2=$\{-1|0\}$, $\{-1|0,1\}$
0=$\{\ |\ \}$, $\{-1|\ \}$, $\{\ |1\}$, $\{-1|1\}$
½=$\{0|1\}$, $\{-1,0|1\}$
1=$\{0|\ \}$, $\{-1,0|\ \}$
2=$\{1|\ \}$,$\{0,1|\ \}$,$\{-1,1|\ \}$, $\{-1,0,1|\ \}$

To prove this, for example that 2 equals for different sets, we first say that 2 = $\{1|\ \}$, and then set x = $\{0,1|\ \}$, and we already know that this value is greater than 0, and then prove 2≥x and x≥2. And then we need to prove the rest of them that way.

Domineering: How does numbers relate to games
$\{L|R\}$ L corresponds to the plays by the left player and R corresponds to the plays by the right player. In domineering we want the Left player to be the player who can only play dominos vertically and the right player to be the player who can only play horizontally.



$\{-1,0|1\}$, $\{\ |-1,0\}$, $\{0|\ \}$ $= \frac{1}{2} + -2 + 1 = -\frac{1}{2}$
Right wins.

Bibliography

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