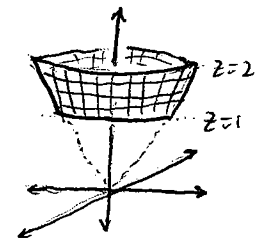
Quiz 7

Name: key

Let S denote the surface in $\mathbf{R}^3$ which is the portion of the cone $z = \sqrt{x^2+y^2}$ between $z = 1$ and $z = 2$ (see diagram).



1. Give a convenient parametrization of S. (That is, express x, y, and z on S as functions of parameters u and v, and give the range of values of u and v corresponding to S.)

Take $u = r$ and $v = \theta$

Then $1 \le u \le 2$, $0 \le v \le 2\pi$, and $\begin{cases} x = u\cos v \\ y = u \sin v \\ z = u \end{cases}$

2. For the parametrization you gave above, find

a. $\mathbf{r}_u = \langle \cos v, \sin v, 1 \rangle$

b. $\mathbf{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

c. $\mathbf{r}_u \times \mathbf{r}_v = \langle -u\cos v, -u\sin v, u \rangle$

d. $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{2}\,u$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \vec{i}(-u\cos v) - \vec{j}(-(-u\sin v)) + \vec{k}(u\cos^2 v + u\sin^2 v) = -u\cos v\,\vec{i} - u\sin v\,\vec{j} + u\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(-u\cos v)^2 + (-u\sin v)^2 + u^2} = \sqrt{u^2(\cos^2 v + \sin^2 v) + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$$

3. Evaluate $\int\int_S z\, dS$.

$$\int_0^{2\pi}\int_1^2 z\,|\vec{r}_u \times \vec{r}_v|\,du\,dv = \int_0^{2\pi}\int_1^2 u(\sqrt{2}u)\,du\,dv = \int_0^{2\pi}\left[\frac{\sqrt{2}u^3}{3}\right]_1^2 dv =$$

$$= \int_0^{2\pi} \frac{\sqrt{2}}{3}[8-1]\,dv = \frac{\sqrt{2}\cdot 7}{3}\cdot 2\pi = \boxed{\frac{14\sqrt{2}\pi}{3}}$$

[Note: you may get $+3\pi$ or -3π for #4, depending on what parametrization you choose for S.]

4. Evaluate $\int\int_S \mathbf{F}\cdot d\mathbf{S}$, where $F = \langle 0, 0, 1\rangle = \mathbf{k}$.

$$\int_0^{2\pi}\int_1^2 \langle 0,0,1\rangle \cdot \langle -u\cos v, -u\sin v, u\rangle\,du\,dv = \int_0^{2\pi}\int_1^2 u\,du\,dv =$$

$$= \int_0^{2\pi}\left[\frac{u^2}{2}\right]_1^2 dv = \int_0^{2\pi}\frac{3}{2}\,dv = 3\pi$$