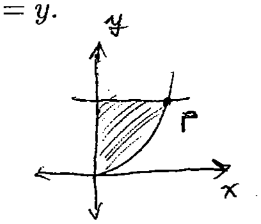
Quiz 5

Name: key

[6] **1.** An object of thickness 1 occupies the region in the first quadrant between the line $y = 1$ and the curve $y = x^2$ (see diagram). Its density is given by $\rho(x, y) = y$. Find the mass of the object. The point P is (1,1).

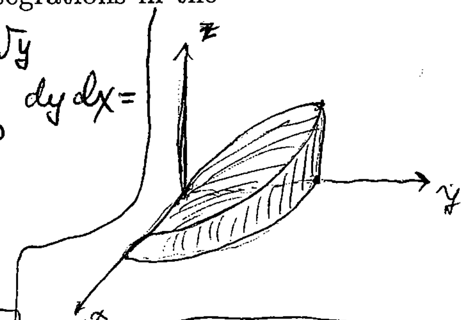


$$\text{Mass} = \int_0^1 \int_{x^2}^1 y \, dy \, dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^2}^{y=1} dx =$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{(x^2)^2}{2} \right) dx = \int_0^1 \left(\frac{1}{2} - \frac{x^4}{2} \right) dx =$$

$$= \left[\frac{x}{2} - \frac{x^5}{10} \right]_{x=0}^{x=1} = \left(\frac{1}{2} - \frac{1}{10} \right) = \frac{4}{10} = \boxed{\frac{2}{5}}.$$

[7] **2.** The region E lies in the first octant of xyz-space, inside $x^2 + y^2 = 4$, and below $z = \sqrt{y}$ (see diagram). Compute $\int\int\int_E z \, dz \, dy \, dx$, doing the integrations in the order indicated.

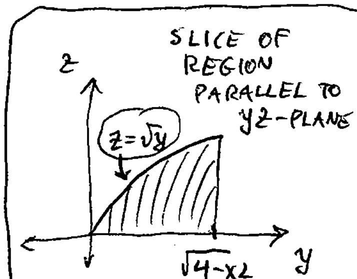


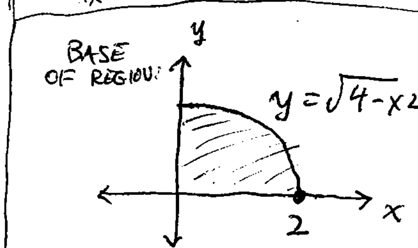
$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{y}} z \, dz \, dy \, dx = \int_0^2 \int_0^{\sqrt{4-x^2}} \left[\frac{z^2}{2} \right]_{z=0}^{z=\sqrt{y}} dy \, dx =$$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} \frac{y}{2} \, dy \, dx = \int_0^2 \left[\frac{y^2}{4} \right]_{y=0}^{y=\sqrt{4-x^2}} dx =$$

$$= \int_0^2 \left(\frac{4-x^2}{4} \right) dx = \int_0^2 \left(1 - \frac{x^2}{4} \right) dx =$$

$$= \left[x - \frac{x^3}{12} \right]_0^2 = 2 - \frac{8}{12} = \boxed{\frac{4}{3}}.$$





[7] **3.** Compute the volume of a sphere of radius 1, using a triple integral in spherical coordinates.

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^3}{3} \sin\varphi \right]_{\rho=0}^{\rho=1} d\varphi \, d\theta =$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{\sin\varphi}{3} \, d\varphi \, d\theta = \int_0^{2\pi} \left[-\frac{\cos\varphi}{3} \right]_{\varphi=0}^{\varphi=\pi} d\theta =$$

$$= \int_0^{2\pi} \left[\left(-\frac{\cos\pi}{3} \right) - \left(-\frac{\cos 0}{3} \right) \right] d\theta = \int_0^{2\pi} \frac{2}{3} \, d\theta = \left[\frac{2\theta}{3} \right]_{\theta=0}^{\theta=2\pi} = \boxed{\frac{4\pi}{3}}.$$