To do this problem easily, you need to realize a fact which I briefly stated in class (but I’m not sure how explicit I made it): Suppose you have a family of extremals \( y(x) \) of a functional emanating from a given point in the \( xy \)-plane; i.e., the family of all solutions of the Euler equation which satisfy \( y(a) = y_0 \), where \( a \) and \( y_0 \) are fixed. Let \( C \) be the envelope curve for this family of extremals; i.e. \( C \) is the curve of points where “two neighboring extremals in the family intersect each other”. Then for each extremal in the family, the conjugate points to \( 0 \) (for that particular extremal) are the points where this extremal touches \( C \).

(A somewhat more precise definition of the envelope is as follows. For each fixed extremal \( \tilde{y} \) with \( \tilde{y}(a) = y_0 \), we will define a point \( P \) on the envelope by taking the limit of intersection points of \( \tilde{y} \) with neighboring extremals. To do this, take any extremal \( \tilde{y} \) with \( \tilde{y}(a) = \tilde{y}_0 \) and \( \tilde{y}'(a) \) close to \( \tilde{y}'(a) \), and look at a point \((x, \tilde{y})\) in the \( xy \)-plane where the graphs of \( \tilde{y} \) and \( \tilde{y} \) intersect. Now change \( \tilde{y} \) so that its graph comes closer and closer to the graph of \( \tilde{y} \); we can do this by making \( \tilde{y}'(a) \) come closer and closer to \( \tilde{y}'(a) \). As we do this, the intersection point \((x, \tilde{y})\) will come closer and closer to a limiting point \( P \). We then say that \( P \) is a point on the envelope of the family. So the envelope is, by definition, the curve composed of all the limiting points \( P \) we get in this way by starting with all possible extremals \( \tilde{y} \).

Now back to the problem. Since \( F = \frac{y}{(y')^2} \) is independent of \( x \), the Euler equation has the first integral \( y' F_{y'} - F = C \), which reduces to the equation

\[
-\frac{3y}{(y')^2} = C.
\]

Solving for \( y' \) and separating variables gives

\[
\int \frac{y'}{\sqrt{y}} \, dx = \int P \, dx,
\]

where \( P \) is a constant. Integrating both sides and solving for \( y \) gives

\[
y = (P x + Q)^2,
\]

where \( P \) and \( Q \) are arbitrary constants. Thus the family of extremals for \( J \) consists of parabolas opening upwards, with their vertices at arbitrary points on the \( x \)-axis and with arbitrarily large steepness. Graphing a few of the parabolas in this family makes clear that their envelope is the \( x \)-axis (not \( x = 0 \) as stated in the text’s hint).

Checking \( F_{y' y'} \) we see that \( F_{y' y'} = 6y/(y')^4 \), which is greater than or equal to zero for \( x \in [0, a] \) for every extremal, and which is strictly positive at each \( x \) for which \( y(x) \neq 0 \).

Now using the conditions \( y(0) = 1 \) and \( y(a) = A \) to find \( P \) and \( Q \), we find that there are two possibilities for \( y \): either

\[
y = y_1 = \left[ \frac{1 - \sqrt{A}}{a} \right]^2 x - 1
\]

or

\[
y = y_2 = \left[ \frac{1 + \sqrt{A}}{a} \right]^2 x - 1.
\]

As explained above, for each of these two extremals, the conjugate point to \( 0 \) (for that particular extremal) is the point where it touches the envelope, which in this case is the \( x \)-axis. It is easy to see that \( y_1 \) touches the \( x \)-axis at a single point, which is outside the interval \([0, a]\). So for \( y_1 \) there are no conjugate points to \( 0 \) in \([0, a]\). Also, since \( y_1 \neq 0 \) on \([0, a]\) then as noted above \( F_{y' y'} > 0 \) on \([0, a]\). Therefore \( y_1 \) satisfies the sufficient conditions for a local minimum given in class, so \( y_1 \) is a local minimum for \( J \) in \( D_1[0, a] \).

For \( y_2 \), on the other hand, there is a conjugate point in \([0, a]\), so we have almost verified that Jacobi’s necessary condition for a local minimum (stated in class and proved in the text; see p. 112) is not satisfied.
by $y_2$. This would then show that $y_2$ is not a local minimum for $J$, since it does not satisfy the necessary condition. The one little difficulty is that as a hypothesis of Jacobi’s necessary condition we must assume that $F_{yy'} > 0$ on $[0, a]$ (see text, p. 112), and that is not the case for $y_2$. One might consider checking the proof of Jacobi’s necessary condition given in the text to see if it could still be made to work under the assumption that $F_{yy'}$ vanishes at one point in $[0, a]$, but I haven’t tried to do this.