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## I. Problem:

Given any random state of a Rubik's Cube, this state can be solved in a minimum number of moves. What is the highest value, n, of this minimum for all states? (I.e. what is the maximal lower bound for an optimal solution of a random cube state.)

(explain that)

## Definitions:

**Metric:** Face turn vs quarter turn metric. The distance between two positions can be defined as the number of quarter turns or face turns necessary to reach one position from the other position. We will use face turn metric.

### Parts of the cube:

A. **Center piece:** The 6 fixed center squares on each face. Each center has 1 sticker on it.
B. **Corner piece:** The 8 smaller cubes each with three stickers.
C. **Edge piece:** The 12 smaller cubes each with two stickers.

### States of the cube:

A. **State:** The arrangement of all faces.
B. **Permutation:** Refers to the manner in which the cubes are relatively positioned.
C. **Orientation:** Refers to the flip of edge pieces and twist of corner pieces but not their position in the cube.

Come back to "flip" and "twist" when needed.

## II. Upper Bound:

The only method for calculating the upper bound, p, for n is to examine the worst case scenario for a known algorithm, either human or computer. A common human algorithm requires four steps, while the computer only uses two to solve the cube.

At each step, many pieces of the puzzle may be disregarded, greatly reducing the number of possible positions to be considered. The number of possible positions at each step is finite, so there must be a maximum number of moves to complete that step which is the worst case scenario for solving that particular step. If we take the worst case scenario for each step, we get an upper bound for n. As an example:

### 4 Steps to Solving with the Human Algorithm

1. **Cross:** involves placing 4 edge pieces correctly, ignoring the rest of the puzzle. Each edge can be placed in 5 turns or less, giving 4*5=20 moves at most for the cross step.

2. **Corner Edge Pairs:** involves placing (simultaneously) a corner and an edge while keeping the Cross intact. Each pair can be solved in at most 12 steps. Again, this must be done four times which gives 4*12 = 48 steps at most for this case. (The www.speedcubing.com algorithm archives)
3. **Orienting the Last Layer:** involves applying one of 57 different algorithms, the longest of which is 11 turns. Thus this step can be solved in at most 11 turns. (The www.speedcubing.com algorithm archives)
4. **Permuting the Last Layer:** involves the application of one of 21 different algorithms, the longest of which is 14 turns. So the last step can be solved in 14 turns or fewer. (The www.speedcubing.com algorithm archives)

From this we reach the conclusion that the worst case scenario for this method of solution would take 20+48+11+14 = 93 turns. So 93 is an upper bound for n.

A computerized algorithm which takes only 2 steps to solve the cube determined (much in the same way as above) that the worst case scenario for its solution is 12 moves for step 1 and 18 moves for step 2. A clever mathematician by the name of Michael Reid proved in 1995 that it is always possible to avoid the 18 move solution in step 2. This yields an upper bound of 12+17 =29 twists for n. (Herbert Kociemba's "Two-Phase Algorithm and God's Algorithm.")

*****partner swap*****

**III. <u>Lower bound for n:</u>**

Computer searches have yielded the lower bound of 20 face turns. Mathematically, this lower bound can be shown to be 17, which we will now do.

**Approach:**

There are a finite number of states the cube can be in. Each face turn yields a finite number of new states. There must therefore be a minimum number of face turns required to reach every possible state. This minimum number is a lower bound for n.

The maximum number of new states for each face turn is 18. This happens because there are 3 possible twists on each of 6 faces for any given state. (demonstration)

| # of turns | Reachable States |
|---|---|
| 0 | 1 |
| 1 | 1+18=19 |
| 2 | 1+18+18^2=343 |
| ... | ... |
| m | Sum from k=0 to m of (18^k) |

This can be improved upon. Turning the same face twice yields a formerly reachable state, creating a duplication in the counting scheme above. This repeated state can be eliminated by disregarding sequential turns of the same face. This gives:

| # of | Reachable States |
|---|---|

| turns | |
| --- | --- |
| 0 | 1 |
| 1 | 1+18=19 |
| 2 | 1+18+18(18-3)=289 |
| 3 | 1+18+18(15)+18(15^2)=4339 |
| ... | ... |
| m | 1+sum from k=0 to m-1 of (18(15^k)) (*) |

This is the revised formula for the maximum number of states reachable in m face turns.

From here we consider the total number of states possible on the cube.

There are 8 corners which may be in any of 8 corner positions. Which yields 8*7*6*5*...*2*1=8! Possible corner permutations.

Also, each corner may be twisted in 3 different ways. This gives 3*3*3*...*3=3^8 possible corner orientations.

There are 12 edges which may be in any of 12 edge positions. Which yields 12*11*...*1= 12! Possible edge permutations.

Further each edge may be flipped in 2 different ways. This gives 2*2*...*2=2^12 possible edge orientations.

The centers do not move, nor are their rotations distinct. Therefore they do not add to the total number of positions.

In total we have at most $8!*3^8*12!*2^{12}$ possible states.

This grand total is not reached because there are restrictions on how the pieces may be positioned and oriented.

**Theorem 1**: It is impossible to swap exactly two pieces.

**Theorem 2**: It is impossible to flip exactly one edge.

**Theorem 3**: Corners must be twisted pair wise in opposite directions. (more explanation while proving)

**<u>Intro to permutations</u>**

To permute a sequence of numbers simply means to rearrange them. Ex: 231 is a permutation of 123.

All permutations can be described as a series of swaps of two elements: 123->213->231. If the number of pair-swaps is even, the permutation is said to have even parity. And if the number of pair-swaps is odd, the permutation is said to be odd. And further no even permutation can be done in an odd number of pair swaps and vice verse. (Theorems 5.4 and 5.5 from *Contemporary Abstract Algebra* By Joseph Gallian).

**Proof of 1:**

Any face turn is an even permuation of pieces. Therefore any sequence of face turns is an even permutation. (Draw pictures to help).

But a swap of exactly two pieces is one pair-swap, which means it is an odd permutation.

Therefore it is impossible to swap exactly 2 pieces.

**Proof of 2:**

Any face turn is an even permutation of edge stickers. Therefore any sequence of face turns is an even permutation of edge stickers.

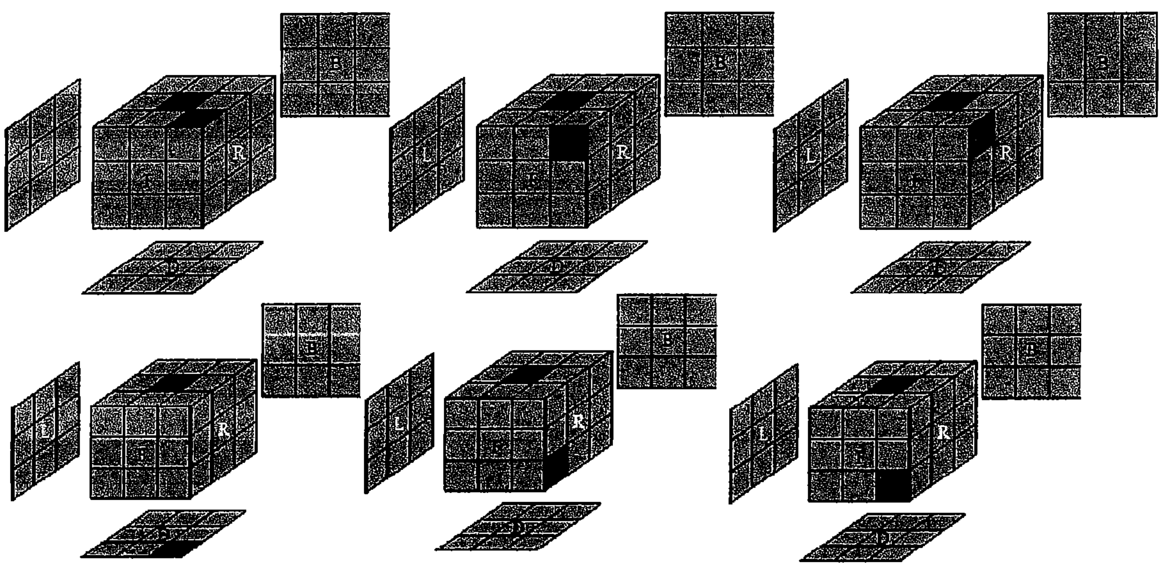
But a swap of exactly two edge sticker (which constitutes the flipping of a single edge piece) is one pair-swap, which means it is an odd permutation.

Therefore it is impossible to flip exactly one edge.

**Proof of 3:**

Since each of the 8 corner cubes contains a sticker of the top or bottom color, and must be a member of the top or bottom layer, we can define twist as follows:

twist = 0 twist = 1 twist = 2



Note that on a solved cube, the twist of each corner is 0, which means the sum of the twists is 0.

Turning the top or bottom face has no affect on the twist of the corner cubes (demonstration).

Turning any of the four remaining faces affects the twists as follows:

Two corners are given twist 2, while two corners are given twist 1 (draw a picture)

Thus the sum of the twist of all corner cubes remains 0 mod 3.

Since no face turn can change the sum of the twists mod 3, no sequence of face turns can change the sum of the twists mod 3.

The rotation of 1 corner clockwise will change the sum of the twists to 2, while a counter clockwise twist will change it to 1. Thus it is impossible to twist exactly one corner.

Likewise twisting two corners in the same direction will have a similar effect and is thus impossible.

Taking these theorems into consideration, it is possible to reduce the calculated total number of possible states of the cube ($8!*3^8*12!*2^{12}$).

Recall, $8!*3^8$ possible corner states. But from theorem 3 after the twist of 7 corners is known, the 8th corner's twist is dependant on the other 7 corner twists. This gives $3^8/3$ total corner twists.

Also, $12!*2^{12}$ possible edge states. But from theorem 2, after the flip of 11 edges is known, the 12th edge flip is dependant on the other 11 edge flips. This gives $2^{12}/2$ possible edge flips.

Further theorem 1 tells us that after 18 of the pieces are placed, the other 2 are dependant on the position of the first 18 (otherwise we would have swapped exactly two pieces). This gives $8!*12!/2$ possible permutations.

In total we have $8!*3^8*12!*2^{12} / (2*2*3) = 43e18$ legal cube positions.

We find from our original equation (*) the fewest number of turns, m, which allows at least 43e18 reachable positions is in fact 17.

Thus 17 is a lower bound for n.

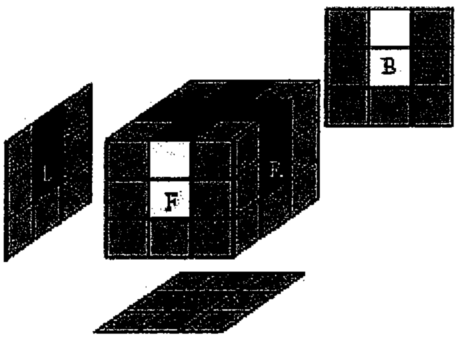
An exhaustive computer search determined that a particular state, called the "superflip", requires no less than 20 turns to solve the Rubik's Cube.

## IV. References

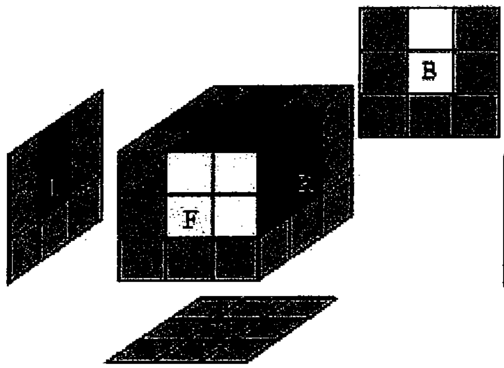

Gallian, Joseph A. Contemporary Abstract Algebra, 6th ed. 2006, *Theorems 5.4 and 5.5 pp102-103.*

Kociemba, Herbert "The Performance of the Two-Phase Algorithm," http://www.kociemba.homepage.t-online.de/performance.htm

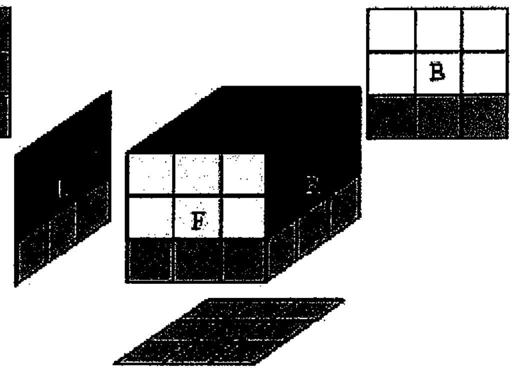
Speedcubing Algorithm Archives, http://www.speedcubing.com/algorithms.html




Step 1
20 turns

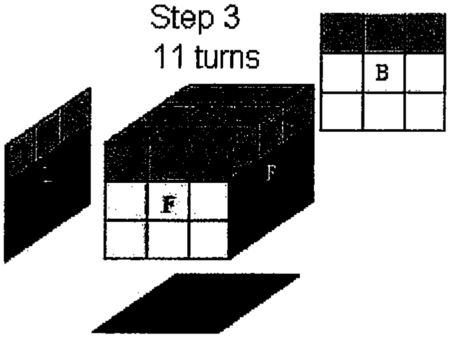



Step 2
12 turns

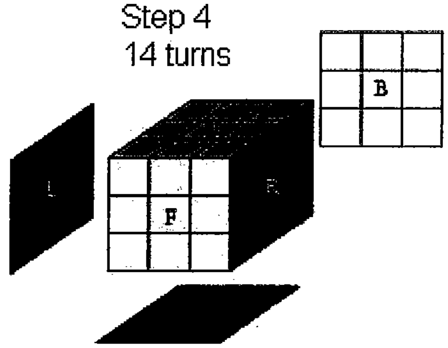



Step 2 X 4
48 turns

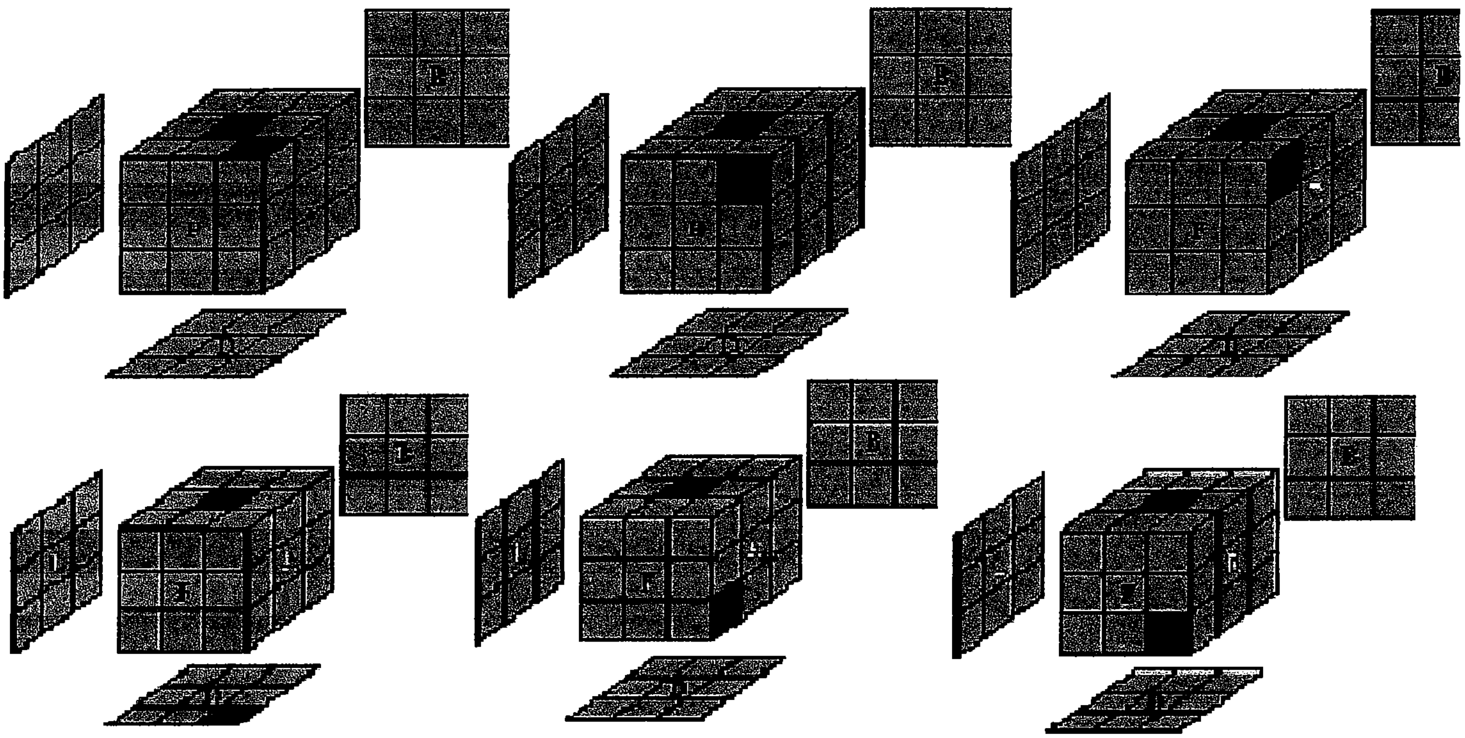
Step 3
11 turns




Step 4
14 turns



Twist = 0 | Twist = 1 | Twist = 2



Twist of each corner after a 1/4 clockwise turn.