Periodic and non-Periodic Tiling of the Plane
And the Unsolved Problem: Can a single
Polygon only tile the plane non-periodically?

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Present dilemma: Can a single polygon tile the plane non-periodically only? In this case we are referring strictly to Euclidean geometry.

Euclid's Axioms:

1. **between any two points there exist a (unique) straight line**
2. **Any line can be extended indefinitely in both directions**
3. **One can draw a circle at any point with any radius**
4. **any two right angles are equal**
5. **(Parallel Postulate) Given a line and a point not on the line there exists a unique parallel line through that point. (Class discussion from Modern geometry notes)**

What we're not talking about: Hyperbolic geometry

1. **Includes first four axioms**
2. **(Hyperbolic parallel postulate) Given a line and a point outside that line there exists at least two parallel lines through that point. (class discussion from modern Geometry notes)**

Definition: Tiling the Plane – a collection of plane figures that fills the plane with no overlaps and no gaps (http://en.wikipedia.org/wiki/Tessellation)

Definition: Periodic Tiling of the Plane – tiling the plane by translation, that is, by shifting the position of the region without rotating or reflecting it. (http://scientium.com/drmatrix/puzzles/progchal.htm)

By these definitions one can show a number of different polygons which can tile the plane periodically, however, it is known that there exists only three regular polygons which can tile the plane periodically, namely, the equilateral triangle, the square and the regular hexagon, which we will now prove.

(Definition: Regular polygon – a polygon in which all the connecting sides and angles are equal. {class discussion})

Proof: Let any given regular polygon in our Euclidean plane consist of a number of sides which is a positive, whole number. Given any regular polygon which tiles the plane periodically, we know the connecting sides of each adjacent polygon must connect in such a way where there are no remaining gaps and no overlaps, and that they tile the plane by translation; the way we do this is to show that the connecting angles divide 360 degrees, or 2pi, to give us a positive integer which demonstrates that there are no gaps or overlaps (E.g. the hexagon vs. the pentagon).

We will begin with the smallest regular polygon (i.e. the one with the fewest amount of sides) and continue to larger polygons and show that the only three that tile the plane periodically are the triangle, square and hexagon.

First, the triangle, which has angles of pi/3 which divides 2pi to give us an integer of 6. Adding a segment we get the square, which has angles of pi/2 which divides 2pi to give us another integer of 4. From here we take the pentagon which has angles of 3pi/5 and divides 2pi to give us a~~n irrational~~ non-whole number 3/10ths (which is the number of pentagons required to construct a periodic tiling – impossible since an integer is required). Next we have the hexagon with angles of 2pi/3 dividing 2pi to give us an integer of 3. Notice however what has happened at this point. By exceeding 2pi/3 we have reduced our total amount of integers for a periodic translation to less than 3, which is the smallest integer for a regular polygon to tile the plane periodically without any gaps or overlaps.

An example of a single tile which can be tile the plane periodically and nonperiodcally. It is based on a 10-gon or decagon.

Definition of a nonperiodic tiling: is one for which there is no translational symmetry.

At the University of Oxford, where he is Rouse Ball Professor of Mathematics, Penrose found small sets of tiles, not of the square type, that force nonperiodicity In 1973 Penrose found a set of six tiles that force nonperiodicity. In 1974 he found a way to reduce them to four. Soon afterward he lowered them to two. (http://scientium.com/drmatrix/puzzles/progchal.htm)

. Let's suppose there are precisely two line segments of length a and b, with a<b. We will propose an inflation process for the prototiles. Under some scaling factor c, a tile of length a becomes a tile of length b. Under the same scaling, a tile of length b splits into a tile of length b followed by a tile of length a. These conditions are equivalent to two equations:

$$ca = b$$

$$cb = b + a$$

Substituting for b in the second, we obtain

$$c^2 a = ca + a$$

Cancelling a on both sides, we have

$$c^2 = c + 1$$

the quadratic equation whose roots are the golden mean and its negative reciprocal

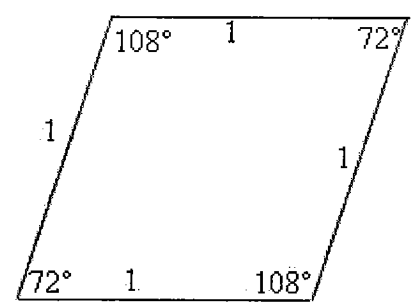
$$c = \frac{1 \pm \sqrt{5}}{2} \doteq 1.618 \text{ or } -0.618$$

You might suppose you need more of the smaller darts, but it is the other way around. You need 1.618 . . . as many kites as darts. In an infinite tiling this proportion is exact. The irrationality of the ratio underlies a proof by Penrose that the tiling is nonperiodic because if it were periodic, the ratio clearly would have to be rational.

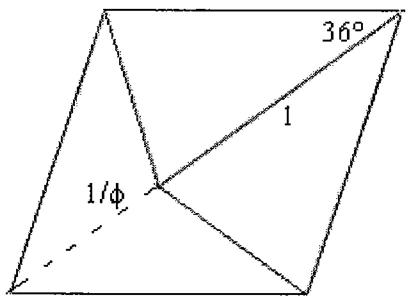
(http://www.math.okstate.edu/mathdept/dynamics/lecnotes/node27.html#figpenrosesubdiv)

In 1973 Sir Roger Penrose (Professor of Oxford) discovered a set of two tiles that tile the plane nonperiodically, but never periodically. These tiles are called Penrose pieces.

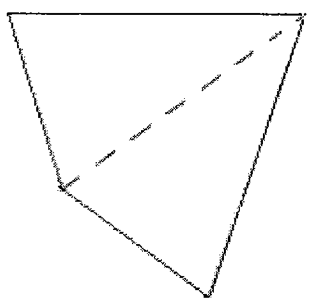
Starting with a rhombus (a quadrilateral in which all 4 sides are equal and opposite sides are parallel) of side length 1, and interior angles of degree 108□ and 72□.



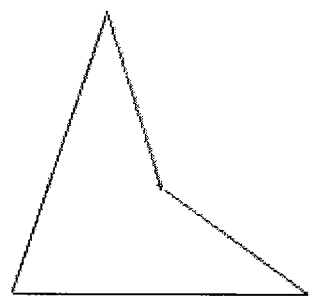
The longer diagonal is then divided so that the longer distance has length 1. It turns out that the length of the remaining distance is 1/□, where □ is the golden ration.



Lines are drawn from the other corner to the end of the diagonal of length 1. This divides the rhombus into two pieces, called a kite, and a dart.

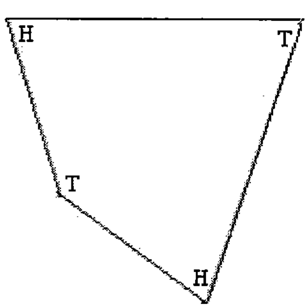


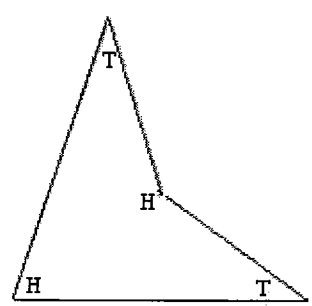
kite



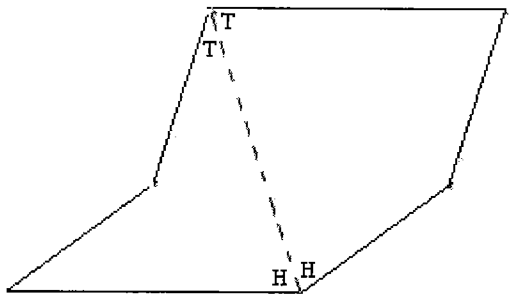
dart

The corners of each piece are marked as H or T (head or tail), and the pieces are combined into tiles with the rule corners with the same letter must go together.

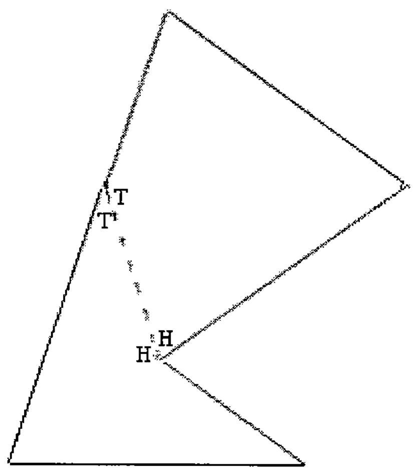




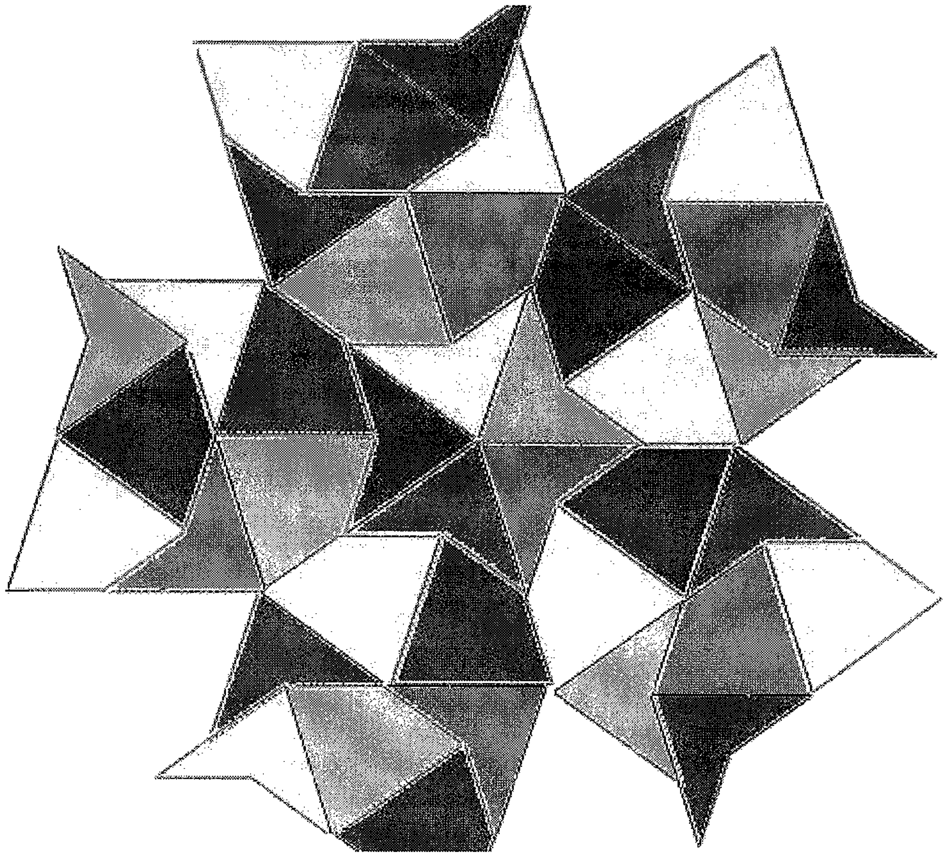
A tile made from attaching sides of length 1



A second tile made from attaching sides of length 1/□



A penrose tiling.



http://cda.mrs.umn.edu/~mcquarrb/SurveyofMath/Resources/Lecture20c.pdf

The number of pieces of the two shapes are (like their areas) in the golden ratio. You might suppose you need more of the smaller darts, but it is the other way around. You need 1.618 . . . as many kites as darts. In an infinite tiling this proportion is exact. The irrationality of the ratio underlies a proof by Penrose that the tiling is nonperiodic because if it were periodic, the ratio clearly would have to be rational.

An infinity of arbitrarily large pairs of shapes, made up of darts and kites, will serve for tiling any infinite pattern.

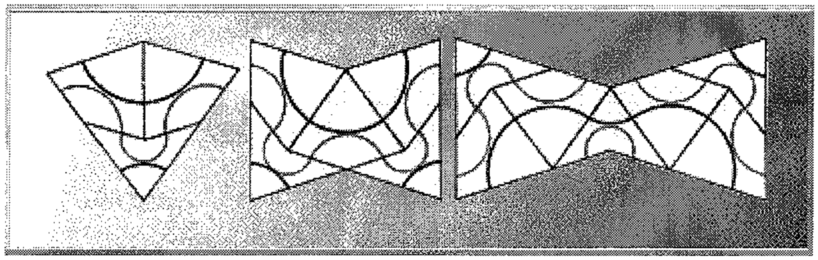


Figure 8

A Penrose pattern is made by starting with darts and kites around one vertex and then expanding radially. Each time you add a piece to an edge, you must choose between a dart and a kite. Sometimes the choice is forced, sometimes it is not. Sometimes either piece fits, but later you may encounter a contradiction (a spot where no piece can be legally added) and be forced to go back and make the other choice. It is a good plan to go around a boundary, placing all the forced pieces first. They cannot lead to a contradiction. You can then experiment with unforced pieces. It is always possible to continue forever. The more you play with the pieces, the more you will become aware of "forcing rules" that increase efficiency. For example, a dart forces two kites in its concavity, creating the ubiquitous ace.

There are many ways to prove that the number of Penrose tilings is uncountable, just as the number of points on a line is. These proofs rest on a surprising phenomenon discovered by Penrose. Conway calls it "inflation" and "deflation." Figure 9 shows the beginning of inflation. Imagine that every dart is cut in half and then all short edges of the original pieces are glued together. The result: a new tiling (shown in heavy dark lines) by larger darts and kites.

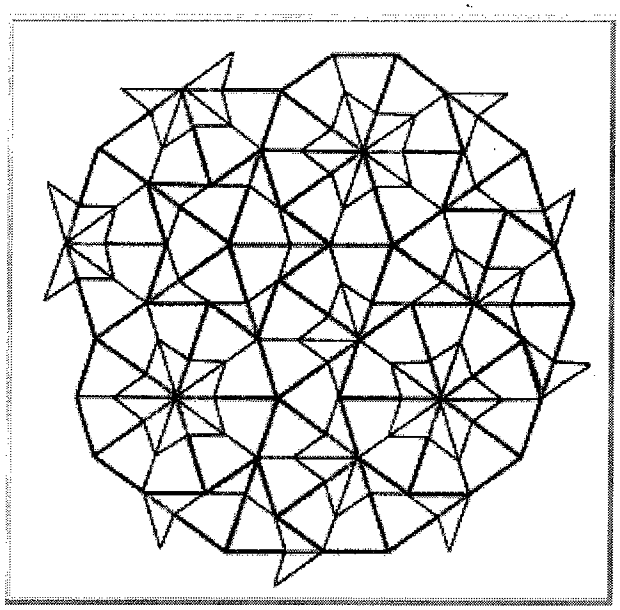


Figure 9: How a pattern is inflated

There is a single polygon which tiles the plane but cannot do so periodically, it has not yet been found.