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The Golden Ratio

Define the golden ratio $\emptyset$ to be a number such that

$\emptyset = 1 + \frac{1}{\emptyset}$ Now multiply by $\emptyset$ to get

$\emptyset^2 = \emptyset + 1$ Note that if you multiply this by $\emptyset^{n-1}$ you get

(A) $\boxed{\emptyset^{n+1} = \emptyset^n + \emptyset^{n-1}}$ This equation (A) will be important later

From $\emptyset^2 = \emptyset + 1$ it follows that $\emptyset^2 - \emptyset - 1 = 0$

By the quadratic formula, $\boxed{\emptyset_1 = \frac{1+\sqrt{5}}{2} \qquad \emptyset_2 = \frac{1-\sqrt{5}}{2}}$

Fibonacci Numbers

The Fibonacci numbers are defined by the recurrence relation

$f_{n+1} = f_n + f_{n-1}$ where $f_0 = 0$ and $f_1 = 1$

The first few Fibonacci numbers are:

$0, 1, 1, 2, 3, 5, 8, 13, \cdots$

<u>Thm</u> The n^{th} Fibonacci number can be written as

$$f_n = \frac{\phi_1^{\,n} - \phi_2^{\,n}}{\sqrt{5}}$$

<u>Proof by mathematical induction</u>

Check that the formula holds for $f_0 = 0$ and $f_1 = 1$

$$f_0 = \frac{\phi_1^{\,0} - \phi_2^{\,0}}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 \checkmark$$ the formula holds for $n=0$

$$f_1 = \frac{\phi_1^{\,1} - \phi_2^{\,1}}{\sqrt{5}} = \frac{\frac{1}{2}(1+\sqrt{5}) - \frac{1}{2}(1-\sqrt{5})}{\sqrt{5}}$$

$$f_1 = \frac{1+\sqrt{5} - 1 + \sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 \checkmark$$ the formula holds for $n=1$

Now assume the formula holds for some f_n and f_{n-1}. Then

$$f_n = \frac{\phi_1^{\,n} - \phi_2^{\,n}}{\sqrt{5}} \quad \text{and} \quad f_{n-1} = \frac{\phi_1^{\,n-1} - \phi_2^{\,n-1}}{\sqrt{5}}$$

By definition $f_{n+1} = f_n + f_{n-1}$ so

$$f_{n+1} = \frac{1}{\sqrt{5}}\left(\phi_1^{\,n} + \phi_1^{\,n-1} - \left(\phi_2^{\,n} + \phi_2^{\,n-1}\right)\right)$$

But (A) says that $\phi^{n+1} = \phi^n + \phi^{n-1}$ for ϕ_1 and ϕ_2

Then $f_{n+1} = \frac{1}{\sqrt{5}}\left(\phi_1^{\,n+1} - \phi_2^{\,n+1}\right)$ so the formula holds for f_{n+1}

$\square$

<u>Claim</u>: $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \phi_1$

<u>Proof</u>:

$$\frac{f_{n+1}}{f_n} = \frac{\frac{\phi_1^{n+1} - \phi_2^{n+1}}{\sqrt{5}}}{\frac{\phi_1^n - \phi_2^n}{\sqrt{5}}} = \frac{\phi_1^{n+1} - \phi_2^{n+1}}{\phi_1^n - \phi_2^n}$$

$$\frac{f_{n+1}}{f_n} = \frac{\phi_1 \phi_1^n - \phi_2 \phi_2^n + \phi_1 \phi_2^n - \phi_1 \phi_2^n}{\phi_1^n - \phi_2^n}$$

$$\frac{f_{n+1}}{f_n} = \frac{\phi_1(\phi_1^n - \phi_2^n) + \phi_2^n(\phi_1 - \phi_2)}{\phi_1^n - \phi_2^n}$$

$$\frac{f_{n+1}}{f_n} = \phi_1 + \frac{\phi_2^n(\phi_1 - \phi_2)}{\phi_1^n - \phi_2^n} = \phi_1 + \frac{\phi_1 - \phi_2}{\left(\frac{\phi_1}{\phi_2}\right)^n - 1}$$

Now $\left|\frac{\phi_1}{\phi_2}\right| > 1 \quad \left(\left|\frac{\phi_1}{\phi_2}\right| \approx 2.6\right)$

Thus $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lim_{n\to\infty} \left(\phi_1 + \frac{\phi_1 - \phi_2}{\left(\frac{\phi_1}{\phi_2}\right)^n - 1}\right)$

since $\left|\frac{\phi_1}{\phi_2}\right| > 1$, $\lim_{n\to\infty} \left|\frac{\phi_1}{\phi_2}\right|^n = \infty$

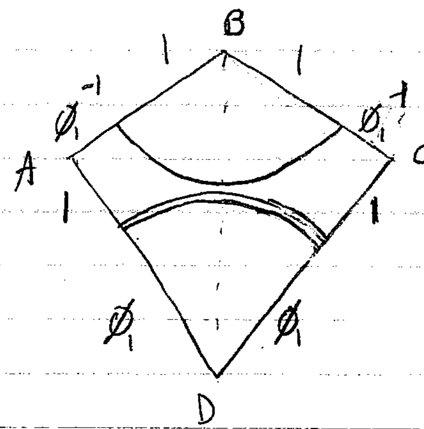
Thus $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \phi_1 + \frac{\text{constant}}{\pm\infty} = \phi_1$

□

Penrose Tiling

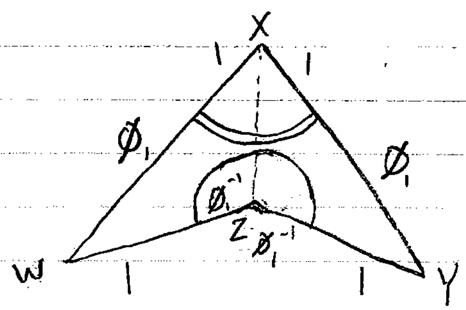
Penrose came up with two tiles that tile the plane aperiodically. That is, there is no region such that translations of that region can tile the plane the same way as the original tiling.

The first Penrose tile is called a kite. It looks like this:



The angles at A, C, and D are 72°
The angle at B is 144°

The second Penrose tile is called a dart. It looks like this:



The angles at W and Y are 36°
The angle at X is 72°
The exterior angle at Z is 144°

The circular arcs are one way of encoding how the tiles are allowed to touch. The only allowed joinings are ones where single-lined arcs join to each other and double-lined arcs join to each other. This restriction is equivalent to making bumps and dents in the sides, but the arcs are easier to see.

Penrose Tiles continued - Inflation

These restrictions guarantee that a process called inflation is always possible in a Penrose tiling of the plane. A proof of this fact is too complex for this presentation, but see the handout for an example.

Inflation essentially takes a Penrose tiling and creates another one with larger individual tiles.
To carry out inflation, cut each dart in half. Now form inflated kites from two old kites and one old dart and inflated darts from one old kite and one old dart.
You now have an inflated Penrose tiling. Inflation can be performed indefinitely.

* Claim: If n_K is the number of kites in a Penrose tiling of the plane and n_D is the number of darts, then $\frac{n_K}{n_D} = \phi$

Proof: Let K' be an inflated kite and D' be an inflated dart.
K and D stand for uninflated kite and dart, respectively.
Now, inflation guarantees that the inflated tiles are composed of:

$$K' = 2K + D \qquad \text{equivalently,} \quad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} K \\ D \end{bmatrix} = \begin{bmatrix} K' \\ D' \end{bmatrix}$$
$$D' = K + D$$

Inflating again yields

$$K'' = 2K' + D' = 2(2K+D) + (K+D) = 5K + 3D$$
$$D'' = K' + D' = (2K+D) + (K+D) = 3K + 2D$$

equivalently, $$\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} K \\ D \end{bmatrix} = \begin{bmatrix} K'' \\ D'' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} K' \\ D' \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} K \\ D \end{bmatrix}$$

Penrose Tiles - Proof of ratio $\frac{n_K}{n_D} = \phi$, continued

Note that these matrices contain sequential Fibonacci numbers. In fact, the original matrix can be written as:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} K \\ D \end{bmatrix} = \begin{bmatrix} f_3 & f_2 \\ f_2 & f_1 \end{bmatrix}\begin{bmatrix} K \\ D \end{bmatrix}$$

Each subsequent inflation multiplies this result by another $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ matrix, since $\begin{bmatrix} K' \\ D' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} K \\ D \end{bmatrix}$

Now $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} = \begin{bmatrix} 2f_{n+1} + f_n & 2f_n + f_{n-1} \\ f_{n+1} + f_n & f_n + f_{n-1} \end{bmatrix}$

$$= \begin{bmatrix} f_{n+2} + f_{n+1} & f_{n+1} + f_n \\ f_{n+1} + f_n & f_n + f_{n-1} \end{bmatrix} = \begin{bmatrix} f_{n+3} & f_{n+2} \\ f_{n+2} & f_{n+1} \end{bmatrix}$$

Thus multiplying a matrix of sequential Fibonacci numbers such as $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ by $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ will yield a matrix with larger sequential Fibonacci numbers.

Thus each inflation yields that an inflated kite is composed of f_m original kites and f_{m-1} original darts, for some $m \in \mathbb{N}$.

As the number of inflations you do approaches ∞, the inflated kite's area also goes to ∞. As the inflated kite's area goes to ∞, the kite is a better and better approximation of the whole plane. Now, the ratio of original kites to original darts in an inflated kite is two sequential Fibonacci numbers, which grow to larger Fibonacci numbers with each inflation. Thus the ratio of n_K/n_D as the number of inflations goes to ∞ is equal to the ratio of two sequential

(see next page)

Penrose Tiles - Proof of $\frac{n_K}{n_D} \doteq \phi_1$ continued

Fibonacci numbers $\left(\frac{F_{n+1}}{F_n}\right)$ as n goes to ∞.

We showed previously that $\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi$

Thus $\frac{n_K}{n_D} \to \phi_1$ as you perform infinitely many inflations.

Thus $\boxed{\frac{n_K}{n_D} = \phi_1}$ for the whole infinite plane

$\square$

Why does this matter? ϕ_1 is irrational, so the ratio $\frac{n_K}{n_D}$ is also irrational. If the tiling were periodic, there would be some region that would be copied infinitely many times. Then the ratio of pieces in this region would be some rational number, since the finite region must contain a finite number of pieces. The ratio in this repeated region would then necessarily be the ratio in the entire plane, so then $\frac{n_K}{n_D}$ would have to be a rational number.

n_K/n_D is not a rational number, so the Penrose tiling must be aperiodic.

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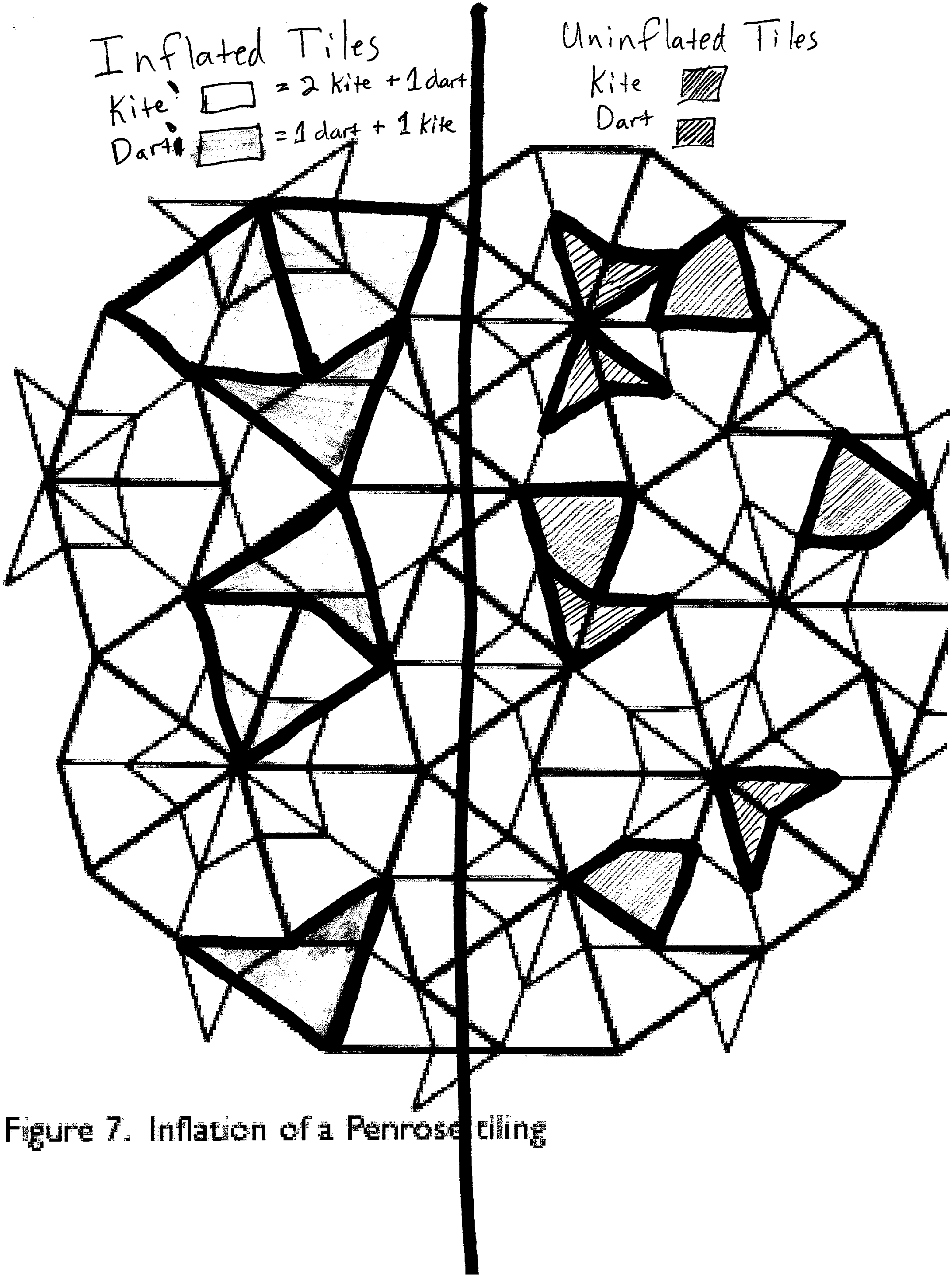

Figure 7. Inflation of a Penrose tiling