Jared Sposato
Jennifer Ford
May 1, 2006

Ramsey's Theory

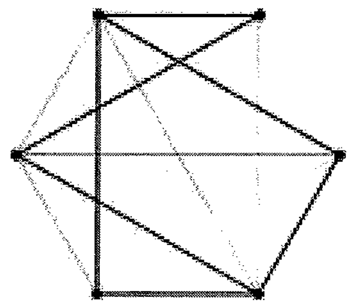
"Ramsey theory is named after Frank Plumpton Ramsey and his eponymous theorem, which he proved in 1928 (it was published in 1930). Ramsey theory is the study of the preservation of properties under set partitions. In other words, given a particular set *S* that has a property *P*, is it true that whenever *S* is partitioned into finitely many subsets, one of the subsets must also have property *P*?" (Landman, 2)

"The **pigeonhole principle** states that if n pigeons are put into m pigeonholes, and if $n > m$, then at least one pigeonhole must contain more than one pigeon." (Wikipedia.org)

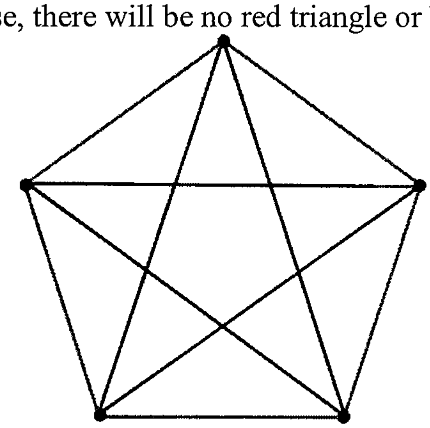
Suppose you are inviting a number of guests to a party, some of whom know each other and some of whom do not know each other. You don't know who knows each other and who doesn't, but you want to make sure that among the guests you invite, there will be either (i) a group of m people, each of whom knows all the others in the group, or (ii) a group of n people, none of whom knows any of the others in the group. What is the minimum number of guests you must invite in order to be sure that either (i) or (ii) happens?

The complete graph with 6 points and 15 edges has each edge colored red or blue. Show that we can find 3 points such that the 3 edges joining them are the same.

By the pigeonhole principle, we are guaranteed that for each person, there are three people that a person has met or three people that person has never met. We now want to show that there are three people with a certain relationship between them, namely, three people who all have met one another, or three people who are mutual strangers. First, assign to each pair of people one of the colors red or blue, with a red "line" connecting two people who have met and a blue "line" connecting two people who are strangers. Hence, we want to show that for any coloring of the lines between people using the colors red and blue, there is either a red triangle or a blue triangle (with people as vertices). Next, pick out one person at the party, say person X. Since there are five other people at the party, by the pigeonhole principle X either knows at least three people, or is a stranger to at least three people. We may assume, without loss of generality, that X knows at least 3 people at the party. Call these people A, B, and C. So far we know that the lines connecting X to each of A, B, and C are red. If there exists a red line between any A, B, and C then we are done, since for example, a red line between A and B would give the red triangle ABX. If the lines connecting A, B, and C are all blue, then ABC is a blue triangle. (Landman, 5,6)



Is 6 the smallest number of party members for the property? To see that you cannot have 5 people, place them all in a circle and assume that each person knows the two people next to them, but no one else, there will be no red triangle or blue triangle. (Landman, 6)



Ramsey's Theorem:

We prove the theorem for the 2-colour case, by induction on $r + s$. It is clear from the definition that for all n, $R(n, 1) = R(1, n) = 1$. This starts the induction. We prove that $R(r, s)$ exists by finding an explicit bound for it. By the inductive hypothesis $R(r - 1, s)$ and $R(r, s - 1)$ exist.

Claim: $R(r, s) \leq R(r - 1, s) + R(r, s - 1)$: Consider a complete graph on $R(r - 1, s) + R(r, s - 1)$ vertices. Pick a vertex v from the graph and consider two subgraphs M and N where a vertex w is in M if and only if (v, w) is blue and is in N otherwise.

Now $|M| \geq R(r - 1, s)$ or $|N| \geq R(r, s - 1)$, again by the pigeonhole principle. In the former case if M has a red K_s then so does the original graph and we are finished. Otherwise M

has a blue K_{r-1} and so M union $\{v\}$ has blue K_r by definition of M. The latter case is analogous.

Thus the claim is true and we have completed the proof for 2 colours. We now prove the result for the general case of c colours. The proof is again by induction, this time on the number of colours c. We have the result for $c = 1$ (trivially) and for $c = 2$ (above). Now let $c > 2$.

Claim: $R(n_1, ..., n_c; c) \leq R(n_1, ..., n_{c-2}, R(n_{c-1}, n_c; 2); c - 1)$

Proof: The right-hand side of the inequality exists by inductive hypothesis. Consider a graph on this many vertices and colour it with c colours. Now 'go colour-blind' and pretend that $c - 1$ and c are the same colour. Thus the graph is now $(c - 1)$-coloured. By the inductive hypothesis, it contains either a K_{ni} monochromatically coloured with colour i for some $1 \leq i \leq (c - 2)$ or a $K_{R(nc-1,nc;2)}$-coloured in the 'blurred colour'. In the former case we are finished. In the latter case, we recover our sight again and see from the definition of $R(n_{c-1}, n_c; 2)$ we must have either a $(c - 1)$-monochrome K_{nc-1} or a c-monochrome K_{nc}. In either case the proof is complete. (Wikipedia.org)

Schur's Theorem:

If N is finitely colored, there exists x, y, and z having the same color such that x + y = z. This may perhaps be considered to be the earliest result in Ramsey theory. It was proved by Schur in 1916.

Proof:

Assume that r colors are used. Let n be such that:

$$n+1 \rightarrow (3)_r$$

An r-coloring of X of [n] induces and r-coloring X* of K_{n+1} on vertex set $\{0,1,\ldots,n\}$ by $X^*(i,j)=X(|i-j|)$, There must exist a monochromatic triangle in K_n, that is $i>j>k$ such that $X^*(i,j)= X^*(j,k)= X^*(i,k)$. Setting $x=i-j$, $y=j-k$, $z=i-k$ gives $X(x)=X(y)=X(z)$ and $x + y = z$

R(m,n) for values of r and s up to 10 are shown below.

m,*n*	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10
3	1	3	6	9	14	18	23	28	36	40–43
4	1	4	9	18	25	35–41	49–61	56–84	69–115	92–149
5	1	5	14	25	43–49	58–87	80–143	101–216	121–316	141–442
6	1	6	18	35–41	58–87	102–165	111–298	127–495	169–780	178–1171
7	1	7	23	49–61	81–143	111–298	205–540	216–1031	232–1713	< 2826
8	1	8	28	56–84	101–216	127–495	216–1031	282–1870	317–3583	< 6090
9	1	9	36	69–115	121–316	169–780	232–1713	317–3583	565–6588	580–12677
10	1	10	40–43	92–149	141–442	178–1171	< 2826	< 6090	580–12677	798–23556

<u>References</u>

Graham, R., Rothschild, B., & Spencer, J. Ramsey Theory. John Wiley & Sons, 1980.

Landman, B.M. & Robertson, A. Ramsey Theory on the Integers. American Mathematical Society, 2000.

"Ramsey's Theorem" From Wikipedia, the free encyclopedia. http://en.wikipedia.org/wiki/Ramsey%27s_theorem

Ramsey's theorem was proved in 1928 (published in 1930). Died @ age 26 in 1930. Theory was subsequently developed by Erdos.

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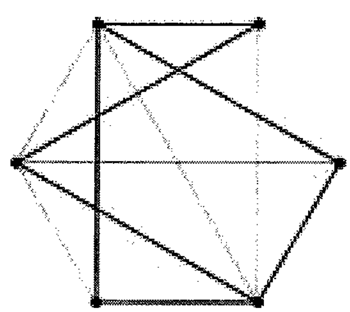

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