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Today, John and I are going to talk about the ability to solve calculus type problems using geometry. First, I'm going to give a couple of examples of how Descartes used circles to find the tangent line to a given curve. John, will then take over and give other examples of how geometry can be used ~~in place of~~ calculus.

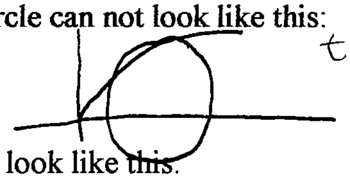
The first example using Descartes method of finding tangents to a curve will be finding the tangent to the curve $y = \sqrt{x}$.

We know a couple of things about tangents.

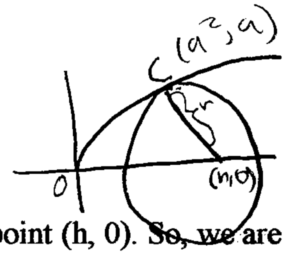
properties of TANGENTS:

1) The line drawn perpendicular to the end point of a radius is a tangent to the circle
2) A tangent to a circle will only contact the circle once

Descartes method approaches the problem of tangents by locating the center of the tangent circle. So, In order for the circle to be tangent to $y = \sqrt{x}$ we know that it can only touch the circle once. So, the circle can not look like this:



So, we know that the circle will look like this.



This requires the use of a circle with center on the x-axis at point (h, 0). So, we are trying to find h so that the circle with center h and radius Ch will meet the curve OC only at the point C.

Where:

r = radius of the circle
(h,0) = point at which r is centered on the x-axis
C= (a^2, a), which is the only point of contact that $y = \sqrt{x}$ makes with the circle.

The circle now has equation Equation of Circle
$(x-h)^2 + y^2 = r^2$
Expand and set equal to zero
$y^2 + x^2 - 2hx + h^2 - r^2 = 0$ and $y = \sqrt{x}$
Now, eliminate y using substitution

we know $x^2 + (1-2h)x + (h^2 + r^2) = 0$

A quadratic with a double root

As stated earlier the circle and curve can only intersect once. So, this means that the above quadratic equation must have a double root. And In order for a quadratic equation $x^2 + px + q$ to have a double root it must be of the form $(x-w)^2$ where w is a root of the quadratic equation and $x^2 + px + q = (x-w)^2$ are equal for all values of x.

So, in order for the circle and curve to be tangent we want a double root at $x = a^2$.

Because, this is a double root the equation now becomes

$x^2 + (1-2h)x + (h^2 + r^2) = (x - a^2)^2$

Expanding the right side

$x^2 + (1-2h)x + (h^2 + r^2) = x^2 - 2a^2x + a^4$

And comparing coefficients, gives us

$(1-2h) = -2a^2$

SO, Solving this equation gives us

$h = a^2 + 1/2$

Therefore, the circle with center $(a^2 + 1/2, 0)$ will be tangent to the graph $y = \sqrt{x}$ at the point (a^2, a).

(a^2, a) $(a^2 + 1/2, 0)$ $\frac{0-a}{a^2+1/2-a^2} = \frac{-a}{1/2} = -2a$

Solving for Radius

We know that the radius through a point C on a circle will be perpendicular to the tangent line of the circle through C. So, radius PC will lie on a line with slope -2a and the tangent line will have slope 1/2a. negative reciprocal

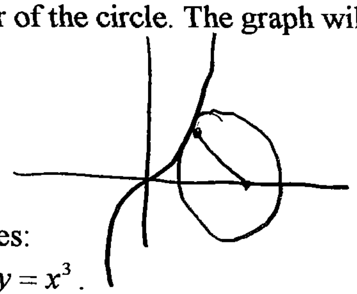
Now, let's ~~see the power~~ do this with of **calculus**. All of the above lines of work can be done in one line. ($\sqrt{x}$) dx = 1/(2$\sqrt{x}$) =1/(2a) when x = a^2.

Let's try another one using Descartes method of finding tangents to a curve.

This shows the inefficiency of Descartes method.

$y = x^3$

Once again, let (h,0) be the center of the circle. The graph will look like this:



The equation of our circle becomes:

$x^2 - 2hx + h^2 + y^2 - r^2 = 0$ and $y = x^3$.

Substituting $y = x^3$ into the equation gives us

Now, we want

$x^6 + x^2 - 2hx + h^2 - r^2 = 0$

Once again we want a double root at point (a, a^3)

Since, the left-hand-side is a 6^{th} degree monic polynomial it must factor as the product of $(x-a)^2$ and a 4^{th} degree monic polynomial.

$x^6 + x^2 - 2hx + h^2 - r^2 = (x-a)^2(x^4 + Bx^3 + Cx^2 + Dx + F)$

Expanding the right-hand-side we get

$$x^6 + x^2 - 2hx + h^2 - r^2 = x^6 + (B-2a)x^5 + (a^2 - 2aB + C)x^4 + (a^2B - 2aC + D)x^3$$
$$+ (a^2C - 2aD + F)x^2 + (a^2D - 2aF)x + a^2F$$

Comparing coefficients gives us the system of equations below:

$(B-2a)=0$	(2)
$(a^2 - 2aB + C)=0$	(3)
$(a^2B - 2aC + D)=0$	(4)
$(a^2C - 2aD + F)=1$	(5)
$(a^2D - 2aF)=-2h$	(6)
$a^2F = h^2 - r^2$	(7)

Using equation (2), we have B=2a and substitute that into equation (3) and we get:

$a^2 - 2a(2a) + C = 0$

So, $C = 3a^2$

Then, substituting B and C into (4) gives us

$a^2(2a) - 2a(3a^2) + D = 0$

So, $D = 4a^3$.

Substituting these values into equation (5) gives us

$a^2(3a^2) - 2a(4a^3) + F = 1$

So, $F = 1 + 5a^4$ and then substitute that into equation (6).

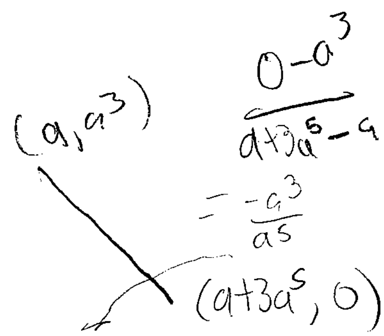
$a^2(4a^3) - 2a(1 + 5a^4) = -2h$

$4a^5 - 2a - 10a^5 = -2h$

$-2a - 6a^5 = -2h$

$h = a + 3a^5$

And the center of the circle will be at $(a + 3a^5, 0)$



Solving for Radius

As before the perpendicular to the curve will have slope $-a^3 / 3a^5 = -1/3a^2$ and thus the slope of the line tangent to the curve $y = x^3$ at $x = a$ will be $3a^2$, the negative reciprocal.

Now, let's do this problem with calculus:

$x^3 dx = 3x^2$ and $a = x$ so the slope of the tangent line is $3a^2$.

Now, John is going to show some more examples.

Johnathan Williams 5/3/06 & Dan Farnham

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NOTES

Honey, Where Should We Sit?

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There are times when, in their haste to solve a particular problem, students (and their instructors) miss an opportunity to notice some interesting mathematics. For example, when calculus students are introduced to the derivatives of inverse trigonometric functions, they frequently run across a classic problem that goes something like this:

There is a 6-foot tall picture on a wall, 2 feet above your eye level. How far away should you sit (on the level floor) in order to maximize the vertical viewing angle θ? (See FIGURE 1.)

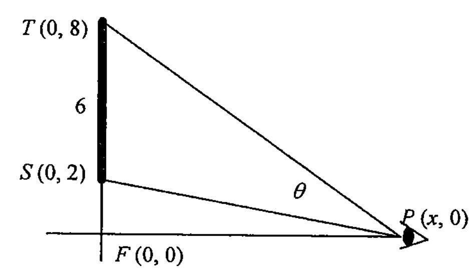

Figure 1 Find where θ is a maximum

This problem can be solved using the standard calculus technique for maximization. First, on the coordinate plane, we could set the top and bottom of the picture at $T(0, 8)$ and $S(0, 2)$, respectively. Then it is easy to show that if your eye is at a point $P(x, 0)$ on the positive x-axis, the viewing angle would be $\theta = \tan^{-1}(8/x) - \tan^{-1}(2/x)$. From the derivative,

$$\frac{d\theta}{dx} = \frac{6(16 - x^2)}{(x^2 + 8^2)(x^2 + 2^2)},$$

you can easily show that the only critical number for $x > 0$ occurs at $x = 4$. Finally, (the part that many students like to skip) the first or second derivative test can provide arguments that θ must be an absolute maximum at $P(4, 0)$.

At this point, many calculus students declare that the greatest viewing angle occurs 4 feet from the wall, express some relief and gratitude for having solved the problem, and move on to the next assignment. In doing so, unfortunately, they miss some fascinating geometry. Notice that, if we let F represent the origin, then at the point P of maximum θ, $PF/FS = 2 = TF/PF$ (FIGURE 2). This makes ΔPFS and

$\triangle TFP$ *similar* right triangles. Thus, the viewing angle is largest at the point P where $\angle FPS \cong \angle FTP$!

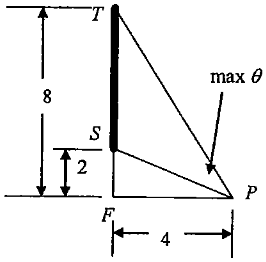

Figure 2 Similar triangles $\triangle PFS$ and $\triangle TFP$

So now a mathematician starts to wonder: is this result just a coincidence (if there is such a thing as a mathematical coincidence)? What if we change the y-coordinates of S and T? How about if, instead of being level, the floor were slanted and P were on a line $y = mx$? (Stewart gives a numerical approach to a variation of this problem [**1**, p. 478].)

Curiously enough, even in these cases the answer is that the viewing angle is a maximum where $\angle FPS \cong \angle FTP$. (This could be a good assignment for a bright student.) In fact, we can generalize even further and consider the case where the floor is curved rather than straight. The result is the following:

THEOREM. *Let $S(0, a)$ and $T(0, b)$ be points on the y-axis with $a < b$, and let $y = f(x)$ be a continuous function on $[0, \infty)$ and, without loss of generality, $f(0) < a$. Then there is point $P(x, f(x))$, $x > 0$, on the graph of f such that the measure of $\angle TPS$ is a maximum. Furthermore, if f is differentiable at P, then $\angle FPS \cong \angle FTP$, where F is the point where the tangent to $f(x)$ at P intersects the y-axis* (FIGURE 3).

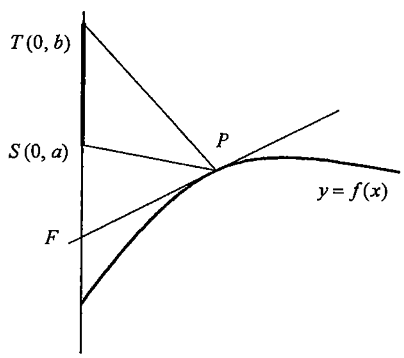

Figure 3 The generalized case

Note: In the original problem P is on the x-axis $y = 0$, and in the variation P is on the line $y = mx$. Both times, the point F is given as the origin. This notation is

consistent with our generalized property sinc
graph of $y = f(x)$, which is simply the graph i
when we refer to a maximum θ, or θ being
ourselves to the domain $(0, \infty)$.

Proof. The property that $\angle FPS \cong \angle FTP$ at
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the time being that a greatest θ exists. It is strai
coordinates $(x,\ f(x))$, then $\angle TPS$ has measu

$$\theta = \tan^{-1}\left(\frac{b - f(x)}{x}\right) + \text{t}$$

Differentiating and simplifying, we can see

$$\frac{d\theta}{dx} = (a - b)\frac{[x^2 + (xf'(x))^2] - [a - (f(x}{[x^2 + (b - f(x)}$$

Since the denominator involves products of sun
neither a nor b, we can see that the denominat
undefined. It follows that at the maximum, the
then,

$$x^2 + (xf'(x))^2 = [a - (f(x) - xf'($$

All we need to do is interpret this in terms o
$f(x)$ at P is $f'(x)$. If we follow the tangent lin
coordinates $(0,\ f(x) - xf'(x))$, as in FIGURE

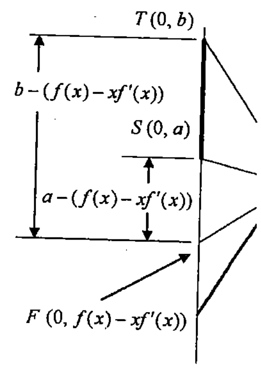

Figure 4 Where the tange

From (1), we see that $PF^2 = SF \cdot TF$, that i

$$\frac{PF}{SF} = \frac{TF}{PF}.$$

Since they share a common angle and have two
that ΔSFP and ΔPFT are similar triangles. The
$\angle FTP$ when P is chosen to make $\angle TPS$ largest.

consistent with our generalized property since, in those cases, the tangent line to the graph of $y = f(x)$, which is simply the graph itself, intersects the y-axis at $(0, 0)$. Also, when we refer to a maximum θ, or θ being maximized, we shall implicitly restrict ourselves to the domain $(0, \infty)$.

Proof. The property that $\angle FPS \cong \angle FTP$ at maximum θ can be proved using standard calculus. Suppose f is differentiable at the maximum angle. We will assume for the time being that a greatest θ exists. It is straightforward to show that, if point P has coordinates $(x,\ f(x))$, then $\angle TPS$ has measure

$$\theta = \tan^{-1}\left(\frac{b - f(x)}{x}\right) + \tan^{-1}\left(\frac{f(x) - a}{x}\right).$$

Differentiating and simplifying, we can see that

$$\frac{d\theta}{dx} = (a - b)\frac{\left[x^2 + (xf'(x))^2\right] - [a - (f(x) - xf'(x))][b - (f(x) - xf'(x))]}{[x^2 + (b - f(x))^2][x^2 + (a - f(x))^2]}.$$

Since the denominator involves products of sums of perfect squares, and since $f(0)$ is neither a nor b, we can see that the denominator is never zero; hence, $d\theta/dx$ is never undefined. It follows that at the maximum, the derivative must be zero. At this point then,

$$x^2 + (xf'(x))^2 = [a - (f(x) - xf'(x))]\,[b - (f(x) - xf'(x))]. \qquad (1)$$

All we need to do is interpret this in terms of lengths. The slope of the tangent to $f(x)$ at P is $f'(x)$. If we follow the tangent line back to the y-axis, we see that F has coordinates $(0,\ f(x) - xf'(x))$, as in FIGURE 4.

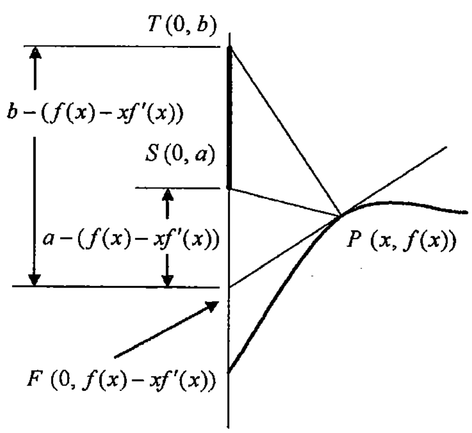

Figure 4 Where the tangent hits the y-axis

From (1), we see that $PF^2 = SF \cdot TF$, that is,

$$\frac{PF}{SF} = \frac{TF}{PF}.$$

Since they share a common angle and have two pairs of proportional sides, it follows that $\triangle SFP$ and $\triangle PFT$ are similar triangles. Therefore, we can conclude that $\angle FPS \cong \angle FTP$ when P is chosen to make $\angle TPS$ largest. ■

Geometric approach Now we turn to some more general questions: Assuming f is continuous, not necessarily differentiable, on $[0, \infty)$, are we guaranteed that there is a point P where the viewing angle is greatest? If there is such a point P, is it necessarily unique or might the maximum angle occur at more than one point on the graph? We can answer these questions by taking a different approach to the problem. Let's leave calculus and its potentially messy computations and turn instead to geometry (with just a pinch of topology).

Recall that, in a circle, the measure of an inscribed angle is one-half that of the intercepted arc [**3**]. A corollary of this property is that every inscribed angle that intercepts the same arc has the same measure. Conversely, given fixed points T and S and an angle θ, the set of all points Q on one side of $\overline{ST}$ satisfying $m(\angle SQT) = \theta$ is a portion of a circle passing through S and T.

Now let's return to our problem. Again, we let S and T represent the top and bottom of our picture. For a fixed positive measure c, consider the set of points Q on the right half-plane such that $m(\angle SQT) = c$. From our discussion above, we can easily see that this level curve is the right-hand portion of a circle passing through S and T (FIGURE 5).

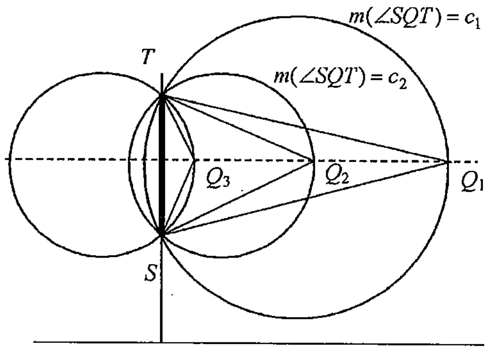

Figure 5 Level curves of constant angles

Moreover, the smaller the value of c, the farther the center of the circle is to the right. For instance, if Q_1, Q_2, and Q_3 are placed on the perpendicular bisector of $\overline{ST}$ as shown in FIGURE 5, it is easy to see that $m(\angle SQ_1T) < m(\angle SQ_2T) < m(\angle SQ_3T)$. Also notice that the regions bounded by $\overline{ST}$ and these circular curves are nested: If $0 < c_1 < c_2$, then the region bounded by $\overline{ST}$ and the curve $m(\angle SQT) = c_2$ is contained in the region bounded by $\overline{ST}$ and $m(\angle SQT) = c_1$.

Now we can answer the questions we posed earlier. Must there be a point P along the graph of $y = f(x)$ at which $m(\angle SPT)$ is a maximum? If so, where is P? The answer to the second question is that P occurs where $y = f(x)$ intersects the circular arc $m(\angle SQT) = c$ for the largest value of c, that is, the leftmost curve $m(\angle SQT) = c$ (FIGURE 6). It is probably obvious that there must be such a point; however, to be safe, we could turn to a little topology. (If this result is obvious, feel free to skip the next paragraph.)

Let G represent the graph of $y = f(x)$. For each positive c, let D_c be the closed bounded region in the right closed half-plane bounded by $\overline{ST}$ and the arc $m(\angle SQT) = c$. Then define G_c to be the intersection of G with D_c. Now consider the nonempty collection $A = \{G_c : G_c \neq \emptyset\}$ of nonempty intersections of G with the sets D_c. The continuity of f implies that G is closed; hence, each G_c is compact. Furthermore, since the D_cs are nested, it follows that the G_cs satisfy the finite intersection

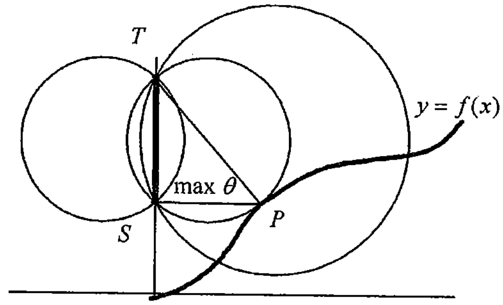

Figure 6 Where θ is maximized

property [2]. Therefore, $\bigcap_{G_c \in A} G_c \neq \emptyset$ and $m(\angle SPT)$ is a maximum at any point P in $\bigcap_{G_c \in A} G_c$.

We can see that this result is consistent with our earlier findings about similar triangles. If the tangent to the circle at P intersects the y-axis at F (FIGURE 7) then, since $\angle SPF$ and $\angle PTF$ intercept the same arc, they are congruent. Consequently, $\triangle SFP$ and $\triangle PFT$ are similar triangles and $PF/SF = TF/PF$, as before.

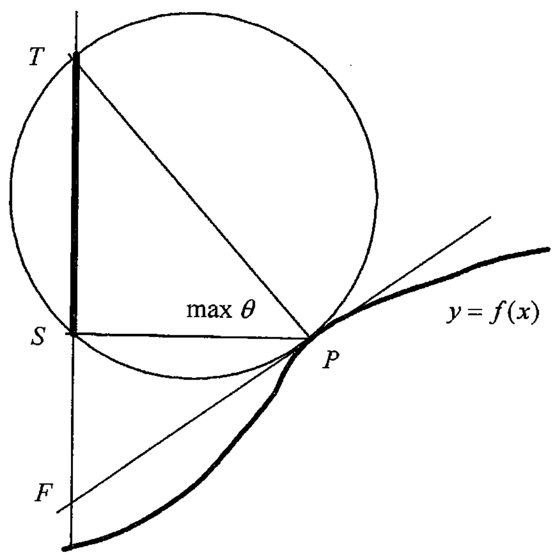

Figure 7 Similar triangles in the general case

This geometric approach allowed us to see, without ugly computations, that there must be a point P on G such that the viewing angle, $m(\angle SPT)$, is maximized. Furthermore, an easy construction allows us to show that, depending upon G, this point of greatest angle may occur at more than one point (FIGURE 8a). In fact, if G moves along a section of one such circular arc, there would be an infinite number of such points (FIGURE 8b).

We now address one final question: How do we construct such a point P? As we showed earlier, sometimes you can find P using possibly cumbersome calculus computations. In the special cases where the graph G is a line, however, we can use the geometry of the situation to physically construct the point of maximum angle using a compass and straightedge. In these situations the smallest circle through S and T that

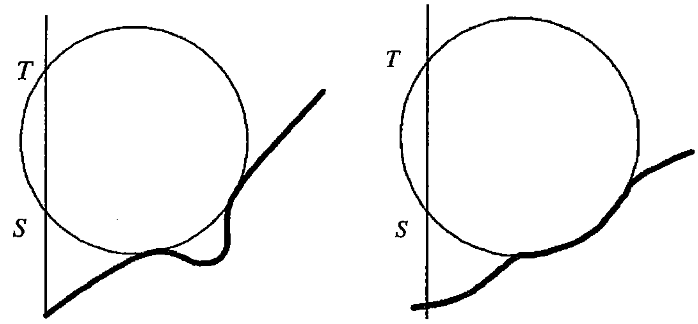

Figure 8 Cases where θ is maximized at multiple points

intersects G must be tangent to G at that point. Thus, all we need to do is find this tangent circle and determine the point P of tangency.

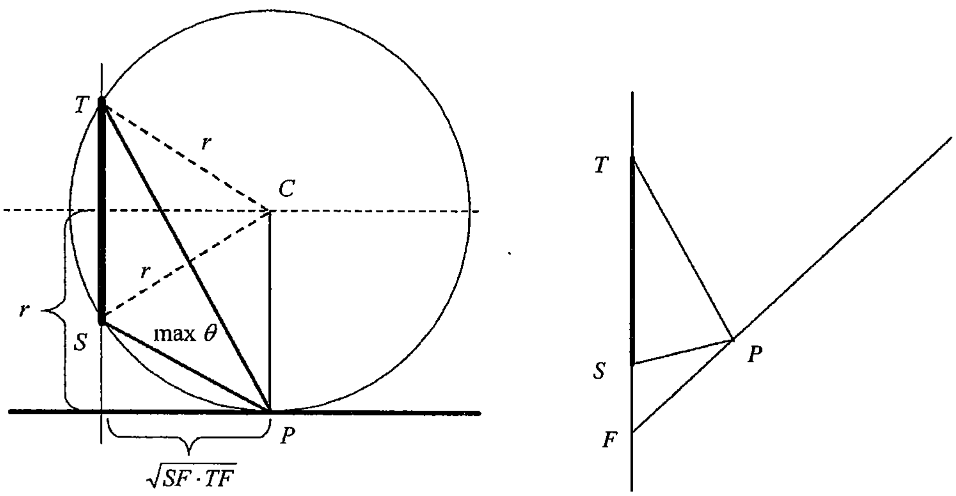

Figure 9 (a) Constructing the point of maximum θ (b) Constructing the slant line solution

This task is especially easy if G is a horizontal line (FIGURE 9a). In this situation, the one we started with, the smallest circle through S and T that intersects G must be tangent to G at that point. Thus, all we need to do is find this tangent circle and determine the point P of tangency. First we find the distance r from the perpendicular bisector of $\overline{ST}$ to G. Next we locate the point C on the right side of this perpendicular bisector that is r units from both S and T. The maximum angle then occurs at the foot P of the perpendicular from C to G. Notice that, from our previous discussion, $PF/SF = TF/PF$; hence, $PF = \sqrt{SF \cdot TF}$, so PF is the geometric mean of SF and TF.

Now that we've constructed the solution for a horizontal line, the solution for the slant line situation becomes easy. At the point of greatest angle measure, we still have the similar triangles, so the distance from P to F is still $PF = \sqrt{SF \cdot TF}$. We constructed this distance in the horizontal line case. All we need to is to construct a circle with center F and radius $\sqrt{SF \cdot TF}$. The desired point P is the intersection of this circle and the slant line (FIGURE 9b).

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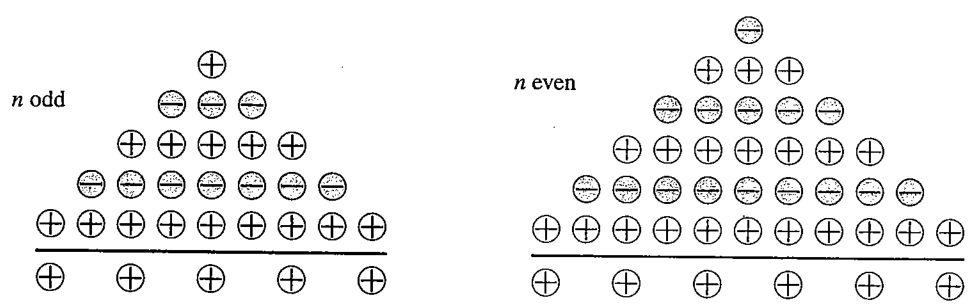
Proof Without Words: Alternating Sums of Odd Numbers

$$\sum_{k=1}^{n}(2k-1)(-1)^{n-k} = n$$



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A Short Proof of Chebychev's Upper Bound

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Examining $\pi(n)$, the number of primes less than or equal to n, is surely one of the most fascinating projects in the long history of mathematics. In 1852, Chebychev [3] proved that there are constants A and B so that, for all natural numbers $n > 1$,

$$\frac{An}{\ln(n)} < \pi(n) < \frac{Bn}{\ln(n)}.$$

Later, in 1896, with arguments of analysis, the Prime Number Theorem was proved, showing that for n sufficiently large, A and B may be taken arbitrarily close to 1. Es-